UNIVERSITY OF LONDON

MSci EXAMINATION 2001

For Internal Students of

Royal Holloway

<u>DO NOT TURN OVER UNTIL TOLD TO BEGIN</u>

PH4211A: STATISTICAL MECHANICS

Time Allowed: TWO AND A HALF hours

Answer THREE QUESTIONS only

No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

Only CASIO fx85WA Calculators ARE permitted

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	$H\ m^{-1}$
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	$F\ m^{-1}$
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	$m\ F^{-1}$
Speed of light in vacuum	c	=	3.00×10^{8}	$m\ s^{-1}$
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	$C\ kg^{-1}$
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	$J\ K^{-1}$
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	$W\ m^{-2}\ K^{-4}$
Gas constant	R	=	8.31	$J\ mol^{-1}\ K^{-1}$
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	$N\ m^{2}\ kg^{-2}$
Acceleration due to gravity	g	=	9.81	$m\ s^{-2}$
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^{3}
One standard atmosphere	P_0	=	1.01×10^{5}	$N\ m^{-2}$

MATHEMATICAL CONSTANTS

$e \cong 2.718$ $\pi \cong 3.142$ $\log_e 10 \cong 2.303$

PART MARKS

1. (a) By considering an isolated system containing a constraint, such as a dividing partition, explain clearly why the equilibrium state, upon removal of the constraint, corresponds to that of maximum entropy. **[6]**

(b) Two systems are brought into contact so that they may exchange thermal energy, mechanical energy and particles. By using the appropriate definitions, show that the equilibrium state corresponds to that in which the temperatures, pressures and chemical potentials of the two systems are equalised. **[8]**

(c) Since the equilibrium state has *maximum* entropy, discuss the implications of the behaviour of the *second derivative* of the entropy with respect to energy of the composite system in terms of the heat capacity. **[6]**

2. When the mean field theory of phase transitions is applied to the ferromagnetic transition in the absence of a magnetic field, the free energy in the vicinity of the transition is given approximately by

$$F = \frac{Nk}{2}(T - T_{\mathrm{c}})m^2 + \frac{Nk}{12}T_{\mathrm{c}}m^4.$$

(a) Explain the structure of this expression and indicate what the various quantities are. **[4]**

(b) By sketching the form of this expression, explain how the transition occurs as the temperature is lowered through T_{c} and discuss the magnitude of the fluctuations in the vicinity of T_{c}. **[4]**

(c) Using the above expression show that in the vicinity of the transition **[5]**

$$m = \pm\sqrt{\frac{3(T_{\mathrm{c}} - T)}{T_{\mathrm{c}}}} \quad \text{for } T < T_{\mathrm{c}}$$
$$= 0 \quad \text{for } T > T_{\mathrm{c}}\,.$$

Sketch this behaviour.

(d) What is the order of this transition? Explain your reasoning. **[3]**

(e) Show that below the transition there is a contribution to the entropy given by **[4]**

$$\frac{3Nk}{2}\frac{T - T_{\mathrm{c}}}{T_{\mathrm{c}}}$$

and that this leads to a jump in the thermal capacity on cooling through the transition of

$$\Delta C = \frac{3}{2}Nk\,.$$

TURN OVER

PART MARKS

3. (a) Outline the arguments by which the Ising hamiltonian

$$\mathcal{H} = -J \sum_{\substack{\text{neighbours} \\ i,j}} S_z^i S_z^j$$

is approximated, in mean field theory, by a local magnetic field

$$\mathbf{b} = \lambda M_z \hat{\mathbf{z}}$$

Where $\hat{\mathbf{z}}$ is the unit vector in the z direction, λ is a constant and M_z is the component of magnetisation in the z direction. **[4]**

(b) The magnetisation of a non-interacting assembly of N spin ½ magnetic moments, μ, is given by

$$M = M_0 \tanh\left(\frac{M_0}{N}\frac{B}{kT}\right)$$

where the saturation magnetisation is $M_0 = N\mu$ and the directions of **M** and the applied magnetic field **B** are parallel.

Show that when the Ising interaction mean field is incorporated, there can be a spontaneous magnetisation given by

$$\frac{M_z}{M_0} = \tanh\left(\frac{M_z}{M_0}\frac{T_c}{T}\right)$$

where $T_c = \lambda M_0^2 / Nk$. What is the interpretation of T_c? Sketch the behaviour of the M_z as a function of temperature and discuss the order of the transition. **[4]**

(c) Now consider the transition in the presence of a *transverse* magnetic field $\mathbf{B} = B_x \hat{\mathbf{x}}$. Write down the magnitude and the direction of the total magnetic field. **[2]**

(d) Hence show that the magnetisation in the z direction is given, within this model, by

$$\frac{M_z}{M_0} = \left(1 + \frac{B_x^2}{\lambda^2 M_z^2}\right)^{-1/2} \tanh\left(\frac{M_z}{M_0}\frac{T_c}{T}\sqrt{1 + \frac{B_x^2}{\lambda^2 M_z^2}}\right).$$ **[4]**

Hint: consider the magnetisation in the direction of the total field.

(e) By rearranging this expression and considering the case where $M_z \to 0$, obtain an expression for the locus of spontaneous M_z in the T–B_x plane and sketch this. **[3]**

(f) Discuss the behaviour of M_z at $T = 0$ as B_x is varied. Why is this referred to as a *quantum* phase transition? **[3]**

PART MARKS

4. Write short notes on three of the following:

(a) conserved and non-conserved order parameters; **[6²/₃]**

(b) the cluster expansion treatment of non-ideal gases; **[6²/₃]**

(c) scaling theory and critical exponents; **[6²/₃]**

(d) the absence of long-range order in one dimensional systems; **[6²/₃]**

(e) The Ising model. **[6²/₃]**

5. (a) The velocity $v(t)$ of a Brownian particle is a randomly varying function of time. Define $G_v(\tau)$, the auto-correlation function for the velocity and explain its physical significance. **[4]**

(b) Show that the mean square displacement of the particle is given by **[4]**

$$\left\langle x^2(t)\right\rangle = 2\int_0^t (t-\tau)G_v(\tau)\,\mathrm{d}\tau\,.$$

(c) What is meant by the correlation time of $G_v(\tau)$? Discuss the limiting behaviour of the mean square displacement for times shorter than, and longer than the correlation time. In particular, show that in the long time limit the mean square displacement is proportional to time whereas in the short time limit the mean square displacement is proportional to time *squared*. **[4]**

(d) Give a physical explanation of the behaviour of the particle in the short and long time limits. **[3]**

(e) Using the above results, show that the diffusion coefficient of the Brownian particle may be expressed in terms of the area under $G_v(\tau)$. **[3]**

(f) Why is this result called a *fluctuation-dissipation* theorem? **[2]**