Royal Holloway

UNIVERSITY OF LONDON

MSci EXAMINATION

ANALYTICAL MECHANICS CP4201A

SUMMER 1999

Time Allowed: TWO AND A HALF HOURS

Answer **THREE** questions only. No credit will be given for attempting a further question.

Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin.

TURN OVER WHEN INSTRUCTED

PH4201A/30

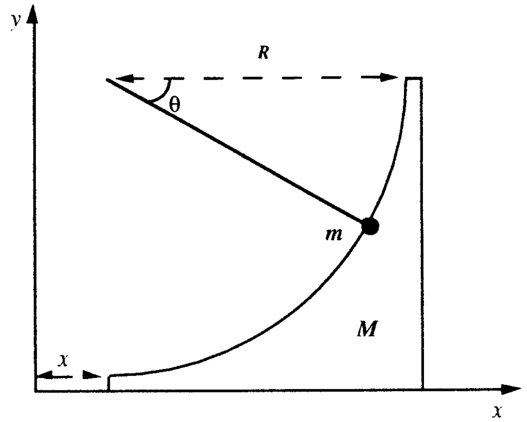
GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	$H\ m^{-1}$
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	$F\ m^{-1}$
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	$m\ F^{-1}$
Speed of light in vacuum	c	=	3.00×10^{8}	$m\ s^{-1}$
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	$C\ kg^{-1}$
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	$J\ K^{-1}$
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	$W\ m^{-2}\ K^{-4}$
Gas constant	R	=	8.31	$J\ mol^{-1}\ K^{-1}$
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	$N\ m^2\ kg^{-2}$
Acceleration due to gravity	g	=	9.81	$m\ s^{-2}$
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^3
One standard atmosphere	P_0	=	1.01×10^{5}	$N\ m^{-2}$

MATHEMATICAL CONSTANTS

$e = 2.718$ $\pi = 3.142$ $\log_e 10 = 2.303$

1. A particle of mass m slides, without friction, down the curved surface of a wedge of mass M which, in turn, is free to slide on a frictionless horizontal surface. The curved surface has a circular cross-section of radius R. (The system is shown in the diagram below). Take the generalised co-ordinates to be x and θ as shown.



Find the Lagrangian for the system. **[5]**

If the particle is released from rest at the top of the wedge ($\theta = 0$) show that, in the subsequent motion, Lagrange's equations reduce to

$$(M+m)\dot{x} = -mR\dot{\theta}\sin\theta \quad \text{and} \quad R\ddot{\theta} + \ddot{x}\sin\theta - g\cos\theta = 0$$ **[11]**

Find the velocity of the particle at the bottom of the wedge. **[4]**

2. A bead of mass m is free to slide on the inside of a bowl given (in cylindrical polar co-ordinates) by $z = \rho^2$. The particle is initially projected horizontally with angular velocity ω at $\rho = a$.

Obtain Lagrange's equations for the system. **[3]**

Obtain Hamilton's equations of motion for the system. **[10]**

Find an expression for the velocity of the particle at any point ρ in the subsequent motion. **[3]**

Find the stationary height of the bead above the bottom of the bowl in the subsequent motion. **[4]**

3. Find the parametric equation of the curve which starts from A(0,0) and ends at a fixed point B such that a disc of radius a and moment of inertia I will roll from rest at A to B in the shortest possible time. [12]

Indicate how the solution would be modified if

(a) it were known that the point B lay on the line $x = b$? [3]

(b) B were fixed but the length of the curve were known? [5]

4. (a) For an action $A = \int_{\tau_1}^{\tau_2} f(L)ds$, where L is the Lagrangian, show that we obtain the usual Euler-Lagrange equation $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$ if L is a constant of the motion. [3]

On a cylinder, co-ordinates are chosen such that the distance between neighbouring points is given by $ds^2 = a^2 d\theta^2 + dz^2$ (where a is a constant).

Find the equation of the geodesic which passes through the points $(0,0)$ and $(\frac{\pi}{2},1)$. [6]

What is the geodesic distance between these points? [2]

(b) Find the geodesic equations in the space-time with metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \quad [6]$$

In particular, for a particle moving in the plane $\theta = \frac{\pi}{2}$, find an expression for $\frac{dr}{dt}$. [3]

5. A particle of mass m which is free to slide on a horizontal surface is connected by an inelastic string of length L which passes through a hole in the surface to a particle of mass M.

If initially the particle on the surface had an angular velocity ω when at a distance L from the hole, obtain Hamilton's equation of motion and show that, in the subsequent motion, if x is the distance from the hole to the particle on the plane;

$$(m+M)\dot{x}^2 = 2MgL + mL^2\omega^2 - 2Mgx - \frac{mL^4\omega^2}{x^2}. \quad [10]$$

Write down the Hamilton-Jacobi equation and, by separation of variables, obtain the resulting equations of motion. Compare these with the above solution. [10]

END