Royal Holloway

UNIVERSITY OF LONDON

MSci EXAMINATION

ANALYTICAL MECHANICS CP4201

SUMMER 2001

Time Allowed: TWO AND A HALF HOURS

Answer **THREE** questions only. No credit will be given for attempting a further question.
Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin.

TURN OVER WHEN INSTRUCTED

PHYSICAL CONSTANTS

PART MARKS

1. (a) Write down Lagrange's equations of motion (with generalised coordinates q_i and generalised velocities $\dot{q}_i$), explaining the significance of the symbols used. **[2]**

Using spherical polar co-ordinates (r, θ, ϕ), find Lagrange's equations of motion of a particle of mass m which is constrained to move on the surface of a smooth sphere of radius R if gravity is the only force acting (i.e. $V = mgR\cos\theta$). **[3]**

Show that the particle satisfies the equation of motion

$$R\ddot{\theta} - \frac{h^2 \cos\theta}{m^2 R^3 \sin^3\theta} - g\sin\theta = 0$$

where $mR^2 \sin^2\theta\dot{\phi} = h =$ constant. **[2]**

If the particle is initially projected horizontally with angular velocity $\dot{\phi} = \Omega$ when the radius vector from the centre of the sphere to the particle makes an angle $\theta = \pi/4$ with the vertical, show that the maximum value $\theta = \theta_m$ in the subsequent motion is given by

$$\frac{R\Omega^2}{4\sin^2\theta_m} + 2g\cos\theta_m = \frac{R\Omega^2}{2} + \sqrt{2}g.$$ **[4]**

(b) Using plane polar co-ordinates, find in terms of an integral, the equation of the curve which encloses a fixed area but which possesses a minimum perimeter. **[7]**

Show that this really is a minimum. **[2]**

2. (a) Using cylindrical co-ordinates (ρ, θ, z), show that the Hamiltonian for a particle of mass m which is constrained to move under gravity on the inside surface of the cone $z = \rho$, and subject to an inverse square law of attraction to the apex of the cone, may be written in the form

$$H = \frac{1}{2m}\left(\frac{p_\rho^{\,2}}{2} + \frac{p_\theta^{\,2}}{\rho^2}\right) - \frac{\mu m}{\sqrt{2}\rho}.$$ **[6]**

where the symbols have their usual meanings.

Write down Hamilton's equations for the above system and find an expression for the velocity of the particle in the subsequent motion, if initially the particle is projected horizontally with angular velocity $\dot{\theta} = \omega$ when $\rho = a$. **[5]**

(b) Find an expression for the curve of fixed length L which starts on the y-axis and ends at (a,0) which, when rotated about the x-axis, encloses a maximum volume. **[7]**

Indicate how the various constants may be evaluated. **[2]**

PART MARKS

3. (a) In the space with metric

$$ds^2 = x^2(dx^2 + dy^2),$$

find the equation relating x and y along a geodesic if it passes through the points $(\cosh 1, 1)$ and $(\cosh 1, -1)$ **[8]**

Find the Hamiltonian for the above and show that the associated Hamilton-Jacobi equation is $\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = x^2$. **[8]**

Solve this equation to find the above geodesic equation. **[4]**

4. (a) A particle of mass m slides without friction (under gravity) inside a cycloid

$$x = a(2\phi + \sin 2\phi); \quad y = a(1 - \cos 2\phi).$$

Obtain the Hamiltonian. **[4]**

Obtain the Hamilton-Jacobi equation, solve this equation by separation of variables and use the result to find an expression for the velocity of the particle. **[9]**

(b) Given a metric in the form

$$ds^2 = -\left(1 + \frac{\gamma}{r}\right)^{-1} dr^2 - r^2 d\theta^2 + \left(1 + \frac{\gamma}{r}\right) dt^2,$$

where γ is a constant, find the conserved quantities associated with the symmetries of this metric. **[7]**

5 The Lagrangian for a relativistic particle of rest mass m_0 moving with velocity v subject to a potential $V = -\mu / r$ may be written

$$L = -m_0 c^2 \sqrt{1 - v^2 / c^2} + \mu / r,$$

show that, in polar co-ordinates, the Hamiltonian is given by

$$H = c\sqrt{p_r^2 + \frac{p_\theta^2}{r^2} + m_0{}^2 c^2} - \mu / r$$

where the symbols have their usual meanings. **[7]**

Obtain the reduced Hamilton-Jacobi equation and, by separation of variables, find a solution. **[9]**

Hence, find the equations of motion. **[4]**

END