UNIVERSITY OF LONDON

MSci EXAMINATION 2002

For Internal Students of

Royal Holloway

<u>DO NOT TURN OVER UNTIL TOLD TO BEGIN</u>

PH4201A: ANALYTICAL MECHANICS

Time Allowed: TWO AND A HALF hours

Answer THREE QUESTIONS only

No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

Only CASIO fx85WA Calculators are permitted

PH4201A/19

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	H m^{-1}
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	F m^{-1}
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	m F^{-1}
Speed of light in vacuum	c	=	3.00×10^{8}	m s^{-1}
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	C kg^{-1}
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	W m^{-2} K^{-4}
Gas constant	R	=	8.31	J mol^{-1} K^{-1}
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	N m^2 kg^{-2}
Acceleration due to gravity	g	=	9.81	m s^{-2}
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^3
One standard atmosphere	P_0	=	1.01×10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

$e \cong 2.718$ $\pi \cong 3.142$ $\log_e 10 \cong 2.303$

1. (a) Using cylindrical co-ordinates (ρ, θ, z), show that the Hamiltonian for a particle of mass *m*, which is constrained to move under gravity on the inside surface of a frictionless bowl whose shape is given by $\rho^2 = z$, and which is subject to an inverse square law of attraction to the bottom of the bowl, may be written in the form

$$H = \frac{1}{2m}\left(\frac{p_\rho{}^2}{(1+4\rho^2)} + \frac{p_\theta^2}{\rho^2}\right) - \frac{\mu m}{\rho\sqrt{1+\rho^2}} \,.$$

Where the symbols have their usual meanings. **[5]**

Write down Hamilton's equations and find an expression for the velocity of the particle in the subsequent motion, if initially the particle is projected horizontally with angular velocity $\dot{\theta} = \omega$ when $\rho = a$. **[5]**

(b) Find the curve of fixed length *L* which starts on the *y*-axis and ends at a fixed point (*b*,0) which, when rotated about the *x*-axis, generates least surface area. **[7]**

Find expressions for the various constants in terms of *L* and *b*. **[3]**

2. (a) Obtain the Hamilton-Jacobi equation appropriate to Fermat's principle of least time for the path of a photon moving in a medium with refractive index

$$n = 1 + \beta y$$ **[5]**

where *y* is a Cartesian co-ordinate and $\beta =$constant.

By separation of variables, find a solution. **[5]**

(b) Given that, in terms of vector and scalar potentials $\boldsymbol{A}$ and ϕ, the Lagrangian for an electromagnetic field is

$$L = \frac{1}{8\pi}\left[\left(\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\right)^2 - (\text{curl}\boldsymbol{A})^2\right],$$

show that ϕ satisfies a wave equation provided $\boldsymbol{A}$ satisfies the relation $\text{div}\boldsymbol{A} = \dfrac{\partial \phi}{\partial t}$. **[10]**

PART MARKS

3. (a) Write down Lagrange's equations of motion for a system subject to α constraints $g_{\alpha}(q_i, t) = 0$, including the forces of constraint. **[2]**

A sphere of radius a with moment of inertia *I*, starts at the top $(\theta = 0)$ and rolls without slipping under gravity on the outer surface of a sphere of radius *R*.

Using plane polar co-ordinates (r, θ) centred on the sphere of radius *R*, find Lagrange's equations of motion for the above system. **[6]**

Show that the spheres lose contact when

$$\frac{2m(R+a)^2\left[1-\cos\theta\right]}{\left[m(R+a)^2 + \dfrac{IR^2}{a^2}\right]} - \cos\theta = 0.$$ **[4]**

(b) Given a metric

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 + \left(1-\frac{2GM}{rc^2}\right)c^2 dt^2,$$

show that the conserved quantities associated with the symmetries of this metric are

$$r^2 \sin^2\theta \frac{d\phi}{ds} = \text{constant}$$

and

$$\left(1-\frac{2GM}{rc^2}\right)\frac{dt}{ds} = \text{constant}.$$ **[6]**

Show also that for motion in the plane $\theta = \pi/2$

$$\dot{r}^2 = a - \left(1-\frac{2GM}{rc^2}\right)\left(1+\frac{b}{r^2}\right)$$ **[2]**

where *a* and *b* are constants.

4. Find the parametric equation of the curve which starts from A(0,0) and ends at a fixed lower point B such that a particle slides under gravity without friction from A to B in the shortest possible time. **[12]**

How would the solution be modified if

(a) it were known that the point B lay on the line $x = b$? **[3]**

(b) B were fixed and the length of the curve were known? **[5]**

5. (a) Using cylindrical co-ordinates (ρ, θ, z) find the equation of the geodesic on the cone $z = k\rho$ which passes through the points

$(1, 0)$ and $(1, -\frac{\pi}{2}(1+k^2)^{\frac{1}{2}})$. **[10]**

(b) A non-relativistic particle of mass m moves in a plane subject to an inverse square law of attraction towards the origin of the co-ordinates. Using plane polar co-ordinates, obtain the Hamilton-Jacobi equation, solve this equation by separation of variables and obtain the equations of motion. **[8]**

Use one of the equations of motion to find an expression for the velocity of the particle. **[2]**

END