Queen Mary
University of London

B. Sc. Examination by course unit 2009

MTH6123 Mathematical Aspects of Cosmology

Duration: 2 hours

Date and time: 04 June 2009, 1000h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): Polnarev

Physical Constants

Gravitational constant	G	6.7×10^{-11} ***N m***^{2} ***kg***$^{-2}$
Speed of light	c	3×10^8 ***m s***$^{-1}$
Solar mass	$M_\odot$	2.0×10^{30}***kg***
Gravitational radius of Sun	$r_{g\odot}$	3 ***km***
Hubble constant	H_0	70 ***km s***$^{-1}$***Mpc***$^{-1}$
Hubble radius	c/H_0	6×10^3 ***Mpc***

1 pc $= 3.1 \times 10^{16}$ ***m***

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ ***and take on the values 1, 2, 3.***
Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ ***and take on the values 0, 1, 2, 3.***
The metric signature $(+ - - -)$ ***is used.***
Partial derivatives are denoted by ",".
Covariant derivatives are denoted by ";".

USEFUL FORMULAS.

Cosmology

$$ds^2 = c^2dt^2 - R^2(t)\left[d\chi^2 + \frac{\sin^2(\sqrt{k}\chi)}{k}(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad \textit{(Robertson–Walker metric)},$$

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda R}{3} \quad \textit{(acceleration equation)}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} \quad \textit{deceleration parameter}$$

$$\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda R^2}{3} \quad \textit{(Friedmann equation)}$$

$$d(\rho c^2 V) = -pdV \quad \textit{(energy conservation equation)}$$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 0.92 \times 10^{-26} \textit{ kg m}^{-3} \quad \textit{(critical density)}$$

$$\Omega_0 = \frac{\rho}{\rho_{crit}} \quad \textit{(density parameter)}$$

General Relativity

Minkowski metric:

$$ds^2 = \eta_{ik}dx^i dx^k = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m,$ ***where*** Γ^i_{kn} ***are Christoffel symbols***

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$u^i = dx^i/ds$ ***is the 4-velocity along the geodesic.***

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \quad \textbf{\textit{where}} \;\; R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.$$

Symmetry properties of the Riemann tensor:

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein equations:

$$R^i_k - \frac{1}{2} \delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where T^i_k ***is the Stress-Energy tensor.***

Gravitational radius:

$r_g = 2GM/c^2 = 3(M/M_\odot)$ ***km***, ***where*** $M_\odot$ ***is the mass of the Sun.***

Section A: Each question carries 10 marks. You should attempt ALL questions.

Question 1 Show that the density parameter Ω_0 (see rubric) is dimensionless. Explain briefly how the value of this parameter can be obtained from direct observations and what is the relation between the dark matter problem and the determination of Ω_0.

Question 2 The Hubble radius is determined as

$$R_H = \frac{c}{H_0},$$

where H_0 is the Hubble parameter. It is given that according to some cosmological model with $k = -1$ the present scale factor, R_0, is twice as large as R_H. Use the Friedman equation to find the density parameter corresponding to such a cosmological model.

Question 3 Assume that the contribution of some low mass Jupiter-like dark objects of mass m to the average density of the Universe is 1% of the critical density. Estimate the average distance d between these objects at the present time. You can assume that the Hubble parameter H_0 is equal to 70km s^{-1}Mpc^{-1} and $m = 10^{-3}M_\odot$. What was the average distance between such objects at the moment corresponding to the redshift $z = 9$?

Question 4 A cosmological model describes the early Universe which contains a perfect fluid with equation of state $p = \alpha\rho c^2$. Using the energy conservation and acceleration equations (see rubric) show that

$$\frac{\rho(R)}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3(1+\alpha)}.$$

Express α in terms of the acceleration parameter q.

Question 5 Consider a spatially flat cosmological model containing dark energy with equation of state $\alpha = -1/2$ and radiation with density parameter $\Omega_{0(r)}$. Using the formula for the dependence of density on scale factor from the previous question, show that the ratio of the total pressure P to the total energy density ρc^2 depends on redshift as

$$\frac{P}{\rho c^2} = \frac{\frac{1}{3}\Omega_{0(r)}(1+z)^4 - \frac{1}{2}(1-\Omega_{0(r)})(1+z)^{3/2}}{\Omega_{0(r)}(1+z)^4 + (1-\Omega_{0(r)})(1+z)^{3/2}}.$$

Find $\Omega_{0(r)}$ if it is given that according to a such model the Universe started to expand with acceleration at $z = 1/4$.

Section B: Each question carries 25 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 6 This question is on Cosmological Models.

Assume that the Universe with $\Lambda = 0$ is open ($k = -1$) and contains only dust. The evolution of the scale factor in this case is given in the following parametric form

$$R(\eta) = \frac{a}{2}(\cosh\eta - 1), \quad t(\eta) = \frac{a}{2c}(\sinh\eta - \eta),$$

where η is a variable which runs from 0 to ∞ and a is some constant.

(a) Using the Friedman equation express a in terms of the Hubble and density parameters. [17]

(b) At some moment of time t_* corresponding to $\eta = \eta_*$ the density of the Universe is equal to ρ_*. Show that the moment of time, t_γ, when the density of the Universe is equal to $\gamma\rho_*$, is

$$t_\gamma \approx t_* \gamma^{-1/3}.$$

Estimate the ratio $\rho(\eta = 10)/\rho(\eta = 20)$. [Hint: Take into account that for such values of η one can approximate $\cosh\eta$ and $\sinh\eta$ by $e^\eta/2 \gg \eta \gg e^{-\eta}/2$.] [8]

Question 7 This question is on the Mathematical Structure of General Relativity.

(a) Give the definition of a covariant tensor of the second rank, A_{ik}, and a mixed tensor of the fourth rank, B^i_{klm}. In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the fourth rank, B^i_{klm}, has only one non-vanishing component, $B^0_{000[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference, x^i, in terms of the transformation matrices $S^l_{m[G]} = \frac{\partial x^l}{\partial x^m_{[G]}}$ and $\tilde{S}^l_{m[G]} = \frac{\partial x^l_{[G]}}{\partial x^m}$. [10]

(b) Using the EFEs and Bianchi identity (see rubric) show that the stress-energy tensor satisfies the conservation law $T^i_{k:i} = 0$. [15]

Question 8 This question is on Mathematical Aspects of Observational Cosmology.

(a) A spherical galaxy of diameter D has redshift z and apparent angular diameter $\Delta\theta$. Using the Robertson-Walker model with co-moving coordinate χ, find the physical distance to this galaxy and show that

$$\Delta\theta = \frac{D\sqrt{k}(1+z)}{R_0 \sin(\sqrt{k}\chi)},$$

where R_0 is a scale factor at the present moment. [7]

(b) Consider radially propagating photons to determine an integral relationship between z and χ. Then assuming that the equation of state parameter $\alpha = 0$ and $k = 0$, use the formula for $\Delta\theta$ from the previous sub-question to find $\Delta\theta$ as a function of z only. Show that the function $\Delta\theta(z)$ is not monotonic even for a spatially flat Universe. Give a very brief qualitative explanation of this effect. Find the value of z at which this function attains its minimum. [18]

Question 9 This question is on Formation of Structure in the Universe.

Consider a dust sphere of average density ρ' in a background flat Universe with $k = \Lambda = 0$. Consider the amplitude of the small density perturbation

$$\triangle(z, M) = \sqrt{\left\langle \frac{\rho'(z, \vec{r}) - \rho(z)}{\rho(z)} \right\rangle^2},$$

where $\rho(R)$ is the average density of the Universe and $\langle\rangle$ means the average over volumes containing mass M. Assume that

$$\triangle(z, M) = \delta(z)F(M),$$

where $F(M)$ is determined by the power spectrum of primordial fluctuations.

(a) Show that $\delta(z)$ as a function of redshift z is the solution of the following equation:

$$\frac{d^2\delta}{dz^2} + \frac{2(1+z)d\delta}{dz} - \frac{3\delta}{2(1+z)^2} = 0.$$

[Hint: Show first that $(R' - R)/R = -\delta/3$.] [15]

(b) Show that the general solution of this equation can be represented in terms of two independent modes, one of which is growing, while the other is decaying. Given that $\delta(z) = 10^{-5}$ at $z = 999$ and $\delta(z) = 1$ at $z = 9$, find $\delta(z)$ at $z = 99$. [10]

End of Paper