Quantum Mechanics 3C26/2002. SOLUTIONS.

All Questions may be attempted. Credit will be given for all correct work done.
[For guidance: A student should aim to answer correctly the equivalent of **THREE** complete questions in the time available].

The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

1. *What is meant in quantum mechanics by the phrase* **Collapse of the Wave Function***?* [2]

A quantum system resides in general in a superposition of eigenstates until a measurement is made. The measurement causes the system to jump irreversibly with a certain probability into one of its eigenstates. The probability of jumping into a particular eigenstate is given by the squared modulus of its probability amplitude. This is known as collapse or reduction of the wave function.

What is the **Copenhagen Interpretation** *of quantum mechanics ?* [4]

The Copenhagen Interpretation, associated with Nils Bohr, places the emphasis on measurement. According to this doctrine, quantum mechanics cannot answer the question of what is happening in detail in an experiment. But if an experiment is carried out with a full specification of the entire apparatus used, the surrounding environment and the precise procedure adopted then quantum mechanics can predict the probability of a particular outcome, i.e. the result of the experiment.

Explain, giving in each case an example which illustrates your explanation, what is meant in quantum theory by (a) **Complementarity***, (b)* **Non-locality.** [8]

(a) Complementarity - Mutually exclusive descriptions (e.g. wave, particle) can be applied to a quantum system but not simultaneously. The wave nature and corpuscular nature of a particle are complementary aspects and never come into conflict in an experimental situation. Example - in the double slit experiment we can either observe an interference pattern between waves or we can

determine trajectories of particles. But one excludes the other - the determination of trajectories destroys the interference pattern; the creation of an interference pattern precludes a precise particle trajectory.

[Could also cite as an example the beam-splitter experiment with semi-silvered mirrors, which is essentially the same physics.]

(b) Non-locality. This is a property of entangled states. Action can be transmitted from one place to affect simultaneously the situation at another arbitrarily distant one.

Classic example is a pair of spin-1/2 particles in a state of total spin S=0. The individual spin components are not defined; all that is known is that they are different; one up, the other down. The particles move apart; measurement of the spin component of one of them immediately fixes the spin component of the other, even though they may be separated by a distance such that no subluminary signal could pass between them. The particles are correlated at all separations. [Could also quote the example given by Einstein, Podolsky and Rosen of a particle separating into two identical fragments which move apart; or that of a pair of polarised photons.]

Discuss the problems that arise when the measurement process is examined in the context of the interaction of a microscopic quantum system and a supposed macroscopic measuring apparatus. **[3]**

The Copenhagen Interpretation implies the existence of two domains, a microscopic quantum domain and a macroscopic, deterministic classical one. Measurements take place at the classical level by a classical measuring device interacting with the quantum system.

But where should the line be drawn between measured object, apparatus, observer? Where does the boundary lie between quantum system subjected to a measurement and the measuring apparatus? If quantum mechanics is a universal theory then all devices should be subject to quantum effects, so when a device interacts with a quantum system it too should pass into a superposition of states, each one corresponding to an eigenstate of the system. The apparatus can only be deemed to have made a measurement when it too collapses into one of its eigenstates. But this requires a second device to perform a measurement on it, and the second a third, and so on, leading to an infinite chain. How can a final collapse be attained to put an end to the chain of measurements?

Why are macroscopic objects not observed to exhibit quantum effects of superposition of states, interference ?

Discuss briefly attempts that have been made to address these problems. **[3]**

These include (brief details expected, not mere headings)

The "quantum potential" approach of Bohm and Hiley, which takes into account the experimental context (environment, apparatus, observer).

The idea that the agent of collapse is a conscious mind (Wigner.)

The many-worlds solution (Everett, Deutsch)

Theories based on alternative histories (Gell-Mann) and decoherence.

[All based on material presented in lectures and notes handed out to all students summarising these lectures.]

2. *The Hamiltonian operator H for a one-dimensional harmonic oscillator of mass m and angular frequency ω is*

$$H = p^2/2m + \tfrac{1}{2}m\omega^2 x^2$$

The creation and annihilation operators a_+ and a_- are defined by

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x); \qquad a_- = a_+^\dagger,$$

where x and p are position and momentum operators satisfying $[x, p] = i\hbar$. From these definitions it may be shown that

$$[a_-, a_+] = 1$$

and

$$H = (a_+ a_- + \tfrac{1}{2})\hbar\omega$$

$$[H, a_+] = a_+\hbar\omega; \qquad [H, a_-] = -a_-\hbar\omega$$

If $H \mid n >= E_n \mid n >$ show that

$$Ha_+ \mid n >= (E_n + \hbar\omega)a_+ \mid n >; \qquad Ha_- \mid n >= (E_n - \hbar\omega)a_- \mid n > .$$

[3]

SOLUTION (From lectures)

$$(Ha_+ - a_+H) \mid n >= a_+ \mid n > \hbar\omega$$
$$Ha_+ \mid n > -a_+E_n \mid n >= a_+ \mid n > \hbar\omega$$

or

$$Ha_+ \mid n >= (E_n + \hbar\omega)a_+ \mid n >$$

Similarly

$$Ha_- \mid n >= (E_n - \hbar\omega)a_- \mid n >$$

What is the interpretation of these equations?

$a_\pm \mid n >$ **is eigenvector of H corresponding to eigenvalue** $E_n \pm \hbar\omega$.

Show further that the lowest eigenvalue $E_0 \geq \frac{1}{2}\hbar\omega$ and that $E_n = (n+\frac{1}{2})\hbar\omega$

To show that $E_n \geq \frac{1}{2}\hbar\omega$: (Bookwork)

$$H \mid n >= E_n \mid n >$$
$$(a_+ a_- + \tfrac{1}{2})\hbar\omega \mid n >= E_n \mid n >$$
$$\hbar\omega < n \mid a_+ a_- \mid n > + \tfrac{1}{2}\hbar\omega < n \mid n >= E_n < n \mid n >$$

Let $a_- \mid n >= C \mid n' >$ where C is a complex constant. Then $< n' \mid a_+ = C^* < n' \mid$ and

$$\hbar\omega \mid C \mid^2 < n' \mid n' > + \tfrac{1}{2}\hbar\omega < n \mid n >= E_n < n \mid n >$$

therefore

$$E_n = \hbar\omega \frac{\mid C \mid^2 < n' \mid n' >}{< n \mid n >} + \tfrac{1}{2}\hbar\omega$$

The scalar products are squared moduli and always positive or zero: Hence

$$E_n \geq \tfrac{1}{2}\hbar\omega.$$

To find energy eigenvalues:

$a_- \mid 0 >= 0$ **because** a_- **is a lowering operator and** $\mid 0 >$ **is the lowest eigenstate. therefore**

$$\hbar\omega a_+ a_- \mid 0 >= (H - \tfrac{1}{2}\hbar\omega) \mid 0 >= 0.$$

$$H \mid 0 >= \tfrac{1}{2}\hbar\omega \mid 0 >$$
$$E_0 = \tfrac{1}{2}\hbar\omega.$$

Now use the raising operator a_+ :

Operate on $\mid 0 >$ with a_+ :

$H a_+ \mid 0 >= (E_0 + \hbar\omega) a_+ \mid 0 >$ **with** $E_0 = \frac{1}{2}\hbar\omega$. **Therefore**

$a_+ \mid 0 >$ **is proportional to the eigenvector** $\mid 1 >$ **corresponding to eigenvalue** $E_1 = (1 + \frac{1}{2})\hbar\omega$.

Repeating the operation we find that

$a_+^2 \mid n >$ **is proportional to the eigenvector** $\mid 2 >$ **corresponding to eigenvalue** $E_2 = (E_1 + \hbar\omega) = (2 + \frac{1}{2})\hbar\omega$.

Repeat n times and we find that $E_n = (n + \frac{1}{2})\hbar\omega$. **[5]**

Matrix element:

If $a_+ \mid n >= X_n \mid n+1 >$, **the constant**

X_n **is**

$$X_n = i(n+1)^{1/2}$$

and also

$$a_- \mid n >= -in^{1/2} \mid n-1 > .$$

hence

$$< m \mid a_+ \mid n >= i(n+1)^{\frac{1}{2}} < m \mid n+1 >= i(n+1)^{\frac{1}{2}} \delta_{mn+1}$$

also, similarly

$$< m \mid a_- \mid n >= -in^{\frac{1}{2}} \delta_{mn-1}$$

Now from the definitions,

$$x = -i\sqrt{\frac{\hbar}{2m\omega}}(a_+ - a_-)$$

Hence

$$< m \mid x \mid n >= \sqrt{\frac{\hbar}{2m\omega}}[\sqrt{n+1}\delta_{mn+1} + \sqrt{n}\delta_{mn-1}]$$ **[6]**

(Down to here, based on lecture notes. The following section is new to the candidates.)

A linear harmonic oscillator of charge q is placed in a weak electric field F directed along the x-axis, which gives rise to a perturbing potential Fqx. Use perturbation theory to determine the energy levels to second order. Show also that an exact solution to the problem yields the same result.

Using the perturbation formula given below with $\lambda V = Fqx$: Diagonal elements of x are zero, so first order correction is zero.

Off diagonal matrix elements are zero unless $m = n+1$ or $m = n-1$ Substitution gives

$$W_n = E_n + q^2F^2\frac{\hbar}{2m\omega}[\frac{n}{+\hbar\omega} + \frac{n+1}{-\hbar\omega}]$$

$$= E_n - \frac{q^2F^2}{2m\omega^2}$$

Exact Solution:

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \tfrac{1}{2}m\omega^2x^2 + qFx$$

Make substitution to complete the square

$$y = x + \frac{qF}{m\omega^2}$$

Then

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dy^2} + \tfrac{1}{2}m\omega^2 y^2 - \frac{q^2F^2}{2m\omega^2}$$

This is the Hamiltonian of an oscillator with the same frequency with a constant term added to the Hamiltonian. The energy levels are just shifted by this constant:

$$W_n = E_n - \frac{q^2F^2}{2m\omega^2}$$

which is the same result as that given by second order perturbation theory. **[6]**

Note: If a quantum system described by a time-independent Hamiltonian H_0 which possesses a known discrete set of non-degenerate eigenvalues E_n with corresponding orthonormal eigenfunctions u_n subjected to a perturbing Hamiltonian λV where λ is a small, real parameter, application of perturbation theory shows that the energy to second order in λ is given by

$$W_n = E_n + < u_n \mid \lambda V \mid u_n > + \sum_{m \neq n} \frac{|< u_m \mid \lambda V \mid u_n >|^2}{E_n - E_m}.$$

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3. A Hermitian operator A has a complete set of orthonormal eigenvectors $\mid n>$ .Show that in the basis $\mid n>$, A is represented by a diagonal matrix with elements $A_{n'n} = a_n \delta_{n'n}$ where a_n is the eigenvalue of A corresponding to eigenvector $\mid n>$. **[2]**

$$A_{n'n} =< n' \mid A \mid n >$$
$$=< n' \mid a_n \mid n >= a_n < n' \mid n >= a_n \delta_{n'n}$$

Hence **A** *is diagonal with elements a_n.*

What is the commutation relation satisfied by J^2 and J_z, and what does it tell us about these two observables? **[2]**

$$[J^2, J_z] = 0.$$

This means that J^2 and J_z are compatible and have simultaneous eigenvectors.

If J_+ and J_- are defined by

$$J_+ = J_x + iJ_y \quad ; \qquad J_- = J_x - iJ_y,$$

show that

$$[J_z, J_+] = \hbar J_+ \quad ; \qquad [J_z, J_-] = -\hbar J_-,$$

and hence that $J_+ \mid j, m >$ and $J_- \mid j, m >$ are proportional to $\mid j, m + 1 >$and $\mid j, m - 1 >$ respectively.

$$J_z(J_x + iJ_y) - (J_x + iJ_y)J_z = J_zJ_x - J_xJ_z + i(J_zJ_y - J_yJ_z)$$

Using the commutation relations this gives

$$i\hbar J_y + i^2\hbar(-J_x)$$
$$= \hbar(J_x + iJ_y) = \hbar J_+.$$

Similarly,

$$J_z(J_x - iJ_y) - (J_x - iJ_y)J_z = J_zJ_x - J_xJ_z - i(J_zJ_y - J_yJ_z)$$

Using the commutation relations this gives

$$i\hbar J_y - i^2\hbar(-J_x)$$
$$= -\hbar(J_x - iJ_y) = \hbar J_-.$$

These relations tell us that $J_+ \mid j, m >$ and $J_- \mid j, m >$ are proportional to $\mid j, m + 1 >$ and $\mid j, m - 1 >$ respectively. **[4]**

(Mainly based on material presented in lectures down to here. What follows is new.)

A particle has total spin quantum number $s = 3/2$. Its spin operator is $\mathbf{S}$. What are the eigenvalues of (a) S_z and (b) S^2 ?

Eigenvalues of S_z are $3/2\hbar, 1/2\hbar, -1/2\hbar, -3/2\hbar$.

Eigenvalue of S^2 is $\frac{3}{2} \times (\frac{3}{2} + 1)\hbar^2$.

Write down the matrices of S_z and S^2 in the basis formed from the normalised eigenvectors $\mid s, m >$ of S_z;

Matrix of S_z is

$$\mathbf{S_z} = \begin{pmatrix} \frac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & \frac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\hbar \end{pmatrix}$$

Matrix of S^2 is $\frac{15}{4}\hbar^2\mathbf{I}$ where $\mathbf{I}$ is the 4×4 unit matrix.

Verify that they commute.

They commute because all 4×4 matrices commute with a multiple of the 4×4 unit matrix. **[3]**

Given that

$$S_\pm \mid s, m >= \hbar\sqrt{s(s+1) - m(m \pm 1)} \mid s, m \pm 1 >$$

find the matrix of S_x in the same basis. Verify that the eigenvalues of S_x are the same as those of S_z. **[9]**

$$S_x = \frac{1}{2}(S_+ + S_-)$$

also

$$S_+ \mid s, m >= \hbar\sqrt{\frac{15}{4} - m(m+1)} \mid s, m+1 >$$

so that

$$< s, m' \mid S_+ \mid s, m >= \hbar\sqrt{\frac{15}{4} - m(m+1)}\delta_{m'm+1}$$

$$< s, m' \mid S_- \mid s, m >= \hbar\sqrt{\frac{15}{4} - m(m-1)}\delta_{m'm-1}$$

Hence the matrices are

$$\mathbf{S_-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{pmatrix}$$

$$\mathbf{S}_+ = \begin{pmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\mathbf{S_x} = \frac{1}{2}(\mathbf{S}_+ + \mathbf{S}_-) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ \sqrt{3}\hbar & 0 & 2\hbar & 0 \\ 0 & 2\hbar & 0 & \sqrt{3}\hbar \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{pmatrix}$$

Eigenvalues λ of S_x are the roots of

$$det \begin{pmatrix} -\lambda & \frac{\sqrt{3}\hbar}{2} & 0 & 0 \\ \frac{\sqrt{3}\hbar}{2} & -\lambda & \hbar & 0 \\ 0 & \hbar & -\lambda & \frac{\sqrt{3}\hbar}{2} \\ 0 & 0 & \frac{\sqrt{3}\hbar}{2} & -\lambda \end{pmatrix} = 0$$

Expanding the determinant,

$$-\lambda(-\lambda[\lambda^2 - \frac{3}{4}\hbar^2] - \hbar(-\lambda\hbar)] - \frac{\sqrt{3}\hbar}{2}[\frac{\sqrt{3}\hbar}{2}(\lambda^2 - \frac{3}{4}\hbar^2] = 0$$

or

$$\lambda^4 - \frac{5}{2}\hbar^2\lambda^2 + \frac{9}{16}\hbar^4 = 0$$

By substitution the values

$$\pm\frac{3}{2}\hbar, \pm\frac{1}{2}\hbar$$

all satisfy this equation.

4. *The Hamiltonian operator H describing a quantum mechanical system in spherical polar co-ordinates has a lowest energy eigenvalue E_0. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of H satisfies*

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0.$$ **[5]**

Use the expansion postulate to expand $F(\mathbf{r})$ in the basis formed by the eigenvectors of H, satisfying

$$H\psi_i = E_i\psi_i; \qquad <\psi_i \mid \psi_j>=\delta_{ij}:$$

$$F = \sum_i a_i \psi_i$$

The expectation value of H in state F is

$$<H>=\int F^* H F d\tau$$

Thus

$$<H>=\int \sum_i a_i^* \psi_i^* H \sum_j a_j \psi_j d\tau$$

$$= \int \sum_{i,j} a_i^* a_j \psi_i^* E_j \psi_j d\tau$$

$$= \sum_{i,j} a_i^* a_j E_i \delta_{ij} = \sum_i \mid a_i \mid^2 E_i$$

Now

$$<F \mid F>=\int F^* F d\tau$$

$$= \int \sum_{i,j} a_i^* a_j \psi_i^* \psi_j d\tau$$

$$= \sum_i \mid a_i \mid^2$$

Let E_0 be the lowest eigenvalue of H, i.e. $E_0 \leq E_i$ for all $i \neq 0$. Then

$$\sum \mid a_i \mid^2 E_i \geq \sum_i \mid a_i \mid^2 E_0 = E_0 \sum_i \mid a_i \mid^2$$

Or

$$\int F^* H F d\tau \geq E_0 \int F^* F d\tau$$

$$E_0 \leq \frac{\int F^* H F d\tau}{\int F^* F d\tau} = E(F)$$

Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit on its value. [2]

Select a trial function $F_t(a, b, c...\mathbf{r})$ depending on variational parameters $a, b, c....$ Vary the parameters to minimise $E(F_t)$. The result is an upper bound on the exact value of E_0.

(Bookwork down to here. The following problem is new but a similar example using a different $V(r)$ was given as a homework problem.)

Use a trial function of the form

$$F(\mathbf{r}, \alpha) = e^{-\alpha r/2}$$

where α is a variational parameter, to investigate the properties of a particle of mass m in a central potential of the form

$$V(r) = V_0(r-3)e^{-r}$$

where V_0 is a positive constant. Show that an upper bound on the ground state energy may be written

$$E(\alpha) = \frac{\hbar^2\alpha^2}{8m} - \frac{3V_0\alpha^4}{(1+\alpha)^4}$$ [5]

where α is a solution of the equation

$$\frac{\hbar^2(1+\alpha)^5}{4m} = 12V_0\alpha^2.$$ [5]

We need to evaluate

$$E(\alpha) = \frac{< F \mid H \mid F >}{< F \mid F >}$$

$$H = \frac{-\hbar^2}{2m}\nabla^2 + V_0(r-b)e^{-\mu r}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$F = e^{-\alpha r/2}$$

independent of θ and ϕ

Thus

$$< F \mid F >= \int e^{-\alpha r} r^2 dr \sin\theta d\theta d\phi = \frac{8\pi}{\alpha^3}$$

$$\nabla^2 e^{-\alpha r/2} = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d}{dr} e^{-\alpha r/2})$$

$$= \frac{1}{r^2}\frac{d}{dr}(r^2(-\alpha/2)e^{-\alpha r/2}$$

$$= \frac{-\alpha}{2r^2}\frac{d}{dr}(r^2 e^{-\alpha r/2})$$

$$= -\frac{\alpha}{2r^2}(2r - \frac{\alpha}{2}r^2)e^{-\alpha r/2}$$

so

$$< T >=< \frac{-\hbar^2}{2m}\nabla^2\Phi >$$

$$= \frac{4\pi\alpha\hbar^2}{4m}\int 2re^{-\alpha r} - \frac{\alpha r^2}{2}e^{-\alpha r} dr$$

with $I_1 = 1/\alpha^2$ and $I_2 = 2/\alpha^3$ this reduces to

$$\frac{\pi\hbar^2}{\alpha m}.$$

$$< V >= 4\pi V_0 \int (r-b) e^{-(\mu+\alpha)r} r^2 dr.$$

$$= 4\pi V_0 [\frac{6}{(\mu+\alpha)^4} - \frac{2b}{(\mu+\alpha)^3}]$$

then

$$E(\alpha) = \frac{\hbar^2\alpha^2}{8m} + \frac{V_0\alpha^3}{(\mu+\alpha)^4}(3 - b\mu - b\alpha)$$

If $b = 3, \mu = 1$ this becomes

$$E(\alpha) = \frac{\hbar^2\alpha^2}{8m} - \frac{3V_0 b\alpha^4}{(\mu+\alpha)^4}$$

To minimise, differentiate w.r.t. α and set derivative to zero:

$$\frac{dE}{d\alpha} = \frac{\hbar^2\alpha}{4m} - \frac{12V_0\alpha^3}{(1+\alpha)^4} + \frac{12V_0\alpha^4}{(1+\alpha)^5}$$

$$= \frac{\hbar^2\alpha}{4m} - \frac{12V_0\alpha^3}{(1+\alpha)^5} = 0$$

or

$$\frac{\hbar^2}{4m} = \frac{12V_0\alpha^2}{(1+\alpha)^5}$$

For just one bound state, $E(\alpha) = 0$. **or**

$$\hbar^2\alpha^2(1+\alpha)^4 = 24mV_0\alpha^4$$

Substituting in the equation for α **gives, after a little algebra,** $\alpha = 1$. **Hence**

$$V_0 = \frac{2\hbar^2}{3m}.$$

Note: If

$$I_n = \int_0^\infty e^{-ax}x^n dx$$

then

$$I_n = \frac{n}{a}I_{(n-1)}$$

and

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}$$

5. *A spin-1/2 particle is placed in an external uniform magnetic field* $\mathbf{B} = B\hat{z}$ *where* $\hat{z}$ *is a unit vector in the z-direction. The Hamiltonian operator is*

$$H = \gamma B S_z$$

where S_z *is the z-component of the spin operator* $\mathbf{S}$. *Find the energy eigenvalues and show that the wave function at time t is*

$$\psi(t) = C_1 e^{-i\omega t/2}\alpha + C_2 e^{i\omega t/2}\beta$$

where C_1 *and* C_2 *are constants and* $\omega = \gamma B$. α *and* β *are eigenvectors of* S_z *corresponding to eigenvalues* $\hbar/2$ *and* $-\hbar/2$ *respectively:*

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

At time $t = 0$ *the particle is in an eigenstate of* S_x *corresponding to eigenvalue* $\hbar/2$

$$\psi(0) = \frac{1}{\sqrt{2}}(\alpha + \beta)$$

At what times will it be in an eigenstate of S_x *corresponding to eigenvalue* $-\hbar/2$

$$\psi(t) = \frac{1}{\sqrt{2}}(\alpha - \beta)$$

Find the expectation value of S_x *at time t.*

$\Big[\mathbf{S} = \frac{\hbar}{2}\sigma$ *where* $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ *and in the basis formed by* α *and* β *the Pauli spin matrices are*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\Big]$$

SOLUTION.

The Hamiltonian $H = \gamma B S_z$ **is independent of time** t**. Solutions to the Time Dependent Schrödinger Equation are of the form**

$$\psi(t) = \psi(0)e^{\frac{-iEt}{\hbar}}$$

where $\psi(0)$ **is a solution of the Time Independent Schrödinger Equation**

$$H\psi(0) = E\psi(0)$$

or

$$\gamma B S_z \psi(0) = E\psi(0)$$

$$S_z\psi(0) = \frac{E}{\gamma B}\psi(0)$$

The eigenvalues of S_z are $\pm\frac{\hbar}{2}$ corresponding to eigenvectors α and β respectively. Thus the energy eigenvalues are

$$E = E_{\pm} = \pm\tfrac{1}{2}\hbar\gamma B$$

The general solution to the TDSE is

$$\psi(t) = C_1\alpha e^{\frac{-iE_+t}{\hbar}} + C_2\beta e^{\frac{iE_-t}{\hbar}}$$

or

$$\begin{aligned}\psi(t) &= C_1\alpha e^{\frac{-i\gamma Bt}{2}} + C_2\beta e^{\frac{i\gamma Bt}{2}} \\ &= C_1\alpha e^{\frac{-i\omega t}{2}} + C_2\beta e^{\frac{i\omega t}{2}}\end{aligned}$$

[7]

where $\omega = \gamma B$.

At $t = 0$,

$$\psi(0) = C_1\alpha + C_2\beta = \frac{1}{\sqrt{2}}(\alpha+\beta)$$

Therefore $C_1 = C_2 = \frac{1}{\sqrt{2}}$.

$$\psi(t) = \frac{1}{\sqrt{2}}(\alpha e^{\frac{-i\omega t}{2}} + \beta e^{\frac{i\omega t}{2}})$$

$$\psi(t) = \frac{1}{\sqrt{2}}e^{\frac{-i\omega t}{2}}(\alpha + \beta e^{i\omega t})$$

We require t such that $e^{i\omega t} = -1$: i.e. $\omega t = \pi, 3\pi, 5\pi, \cdots$ or

$$\omega t = (2N+1)\pi, N = 0, 1, 2, \cdots$$

[5]

(As far as here, material presented in lectures. The next part is new.)

EXPECTATION VALUE OF S_x :

$S_x = \frac{\hbar}{2}\sigma_x$

$$< S_x >= \frac{\hbar}{4}\begin{pmatrix} e^{\frac{1}{2}i\omega t} & e^{-\frac{1}{2}i\omega t}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t}\end{pmatrix}$$

$$< S_x >= \frac{\hbar}{4}\begin{pmatrix} e^{\frac{1}{2}i\omega t} & e^{-\frac{1}{2}i\omega t}\end{pmatrix}\begin{pmatrix} e^{\frac{1}{2}i\omega t} \\ e^{-\frac{1}{2}i\omega t}\end{pmatrix}$$

$$= \frac{\hbar}{4}(e^{i\omega t} + e^{-i\omega t}) = \frac{\hbar}{2}\cos\omega t.$$

The uncertainty in S_x, ΔS_x, is given by the absolute value of

$$\sqrt{(< S_x^2 > - < S_x >^2)}$$

$$S_x^2 = \frac{\hbar^2}{4}\mathbf{I}$$

where I is the 2×2 unit matrix. Hence

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4}(1 - \cos^2\omega t)}$$

$$= \frac{\hbar}{2} \mid \sin\omega t \mid .$$

This is zero when ωt is an integer multiple of π, i.e. the system is in an eigenstate of S_x at these times, in accordance with the first part of the question. [8]

END OF PAPER.