**All Questions may be attempted.** Credit will be given for all correct work done.

[For guidance: A student should aim to answer correctly the equivalent of **THREE** complete questions in the time available].

The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

1. What is meant in quantum mechanics by the phrase **Collapse of the Wave Function**? **[2]**

   What is the **Copenhagen Interpretation** of quantum mechanics ? **[4]**

   Explain, giving in each case an example which illustrates your explanation, what is meant in quantum theory by (a) **Complementarity**, (b) **Non-locality.** **[8]**

   Discuss the problems that arise when the measurement process is examined in the context of the interaction of a microscopic quantum system and a supposed macroscopic measuring apparatus. **[3]**

   Discuss briefly attempts that have been made to address these problems. **[3]**

2. The Hamiltonian operator $H$ for a one-dimensional harmonic oscillator of mass $m$ and angular frequency $\omega$ is

$$H = p^2/2m + \tfrac{1}{2}m\omega^2 x^2$$

The operators $a_+$ and $a_-$ are defined by

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x); \qquad a_- = a_+^\dagger,$$

where $x$ and $p$ are position and momentum operators satisfying $[x, p] = i\hbar$.

You may assume without proof that

$[a_-, a_+] = 1, \qquad H = (a_+a_- + \frac{1}{2})\hbar\omega,$ and $\qquad [H, a_\pm] = \pm a_\pm \hbar\omega.$

Using the notation $H \mid n >= E_n \mid n >$ show that

$$Ha_+ \mid n >= (E_n + \hbar\omega) \mid n >; \qquad Ha_- \mid n >= (E_n - \hbar\omega) \mid n > .$$

What is the interpretation of these equations? **[3]**

Show that

(a) the lowest eigenvalue $E_0 \geq \frac{1}{2}\hbar\omega$

and

(b) $E_n = (n + \frac{1}{2})\hbar\omega$. **[5]**

Using the results

$$a_+ \mid n >= i(n+1)^{1/2} \mid n+1 >$$

and

$$a_- \mid n >= in^{1/2} \mid n-1 >,$$

show that

$$< m \mid x \mid n >= \sqrt{\frac{\hbar}{2m\omega}}[\sqrt{n+1}\ \delta_{mn+1} + \sqrt{n}\ \delta_{mn-1}]$$ **[6]**

A linear harmonic oscillator carrying charge $q$ is placed in a weak electric field $F$ directed along the x-axis, which gives rise to a perturbing potential $Fqx$. Use perturbation theory to determine the energy levels to second order. Show also that an exact solution to the problem yields the same result. **[6]**

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Note: If a quantum system, described by a time-independent Hamiltonian $H_0$ which possesses a known discrete set of non-degenerate eigenvalues $E_n$ with corresponding orthonormal eigenfunctions $u_n$ is subjected to a perturbing Hamiltonian $\lambda V$ where $\lambda$ is a small, real parameter, application of perturbation theory shows that the energy to second order in $\lambda$ is given by

$$W_n = E_n + < u_n \mid \lambda V \mid u_n > + \sum_{m \neq n} \frac{|< u_m \mid \lambda V \mid u_n >|^2}{E_n - E_m}.$$

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3. A Hermitian operator $A$ has a complete set of orthonormal eigenvectors $| n >$ . Show that in the basis $| n >$, $A$ is represented by a diagonal matrix with elements $A_{n'n} = a_n \delta_{n'n}$ where $a_n$ is the eigenvalue of $A$ corresponding to eigenvector $| n >$ . **[2]**

If $\mathbf{J}$ is a quantum mechanical angular momentum operator, write down the commutation relation between $J^2$ and $J_z$. What does it tell us about these two observables? **[2]**

If $J_+$ and $J_-$ are defined by

$$J_+ = J_x + iJ_y \quad ; \qquad J_- = J_x - iJ_y,$$

show that

$$[J_z, J_+] = \hbar J_+ \quad ; \qquad [J_z, J_-] = -\hbar J_-,$$

and hence, if

$$J_z \mid j, m >= m\hbar \mid j, m >$$

that $J_+ \mid j, m >$ and $J_- \mid j, m >$ are proportional to $\mid j, m+1 >$ and $\mid j, m-1 >$ respectively. **[4]**

A particle has total spin quantum number $s = 3/2$. Its spin operator is $\mathbf{S}$. What are the eigenvalues of (a) $S_z$ and (b) $S^2$ ?

Write down the matrices of $S_z$ and $S^2$ in the basis formed from the normalised eigenvectors $\mid s, m >$ of $S_z$.

Verify that they commute. **[3]**

Given that

$$S_\pm \mid s, m >= \hbar\sqrt{s(s+1) - m(m \pm 1)} \;\; \mid s, m \pm 1 >$$

find the matrix of $S_x$ in the same basis.

Verify that the eigenvalues of $S_x$ are the same as those of $S_z$. **[9]**

4. The Hamiltonian operator $H$ describing a quantum mechanical system in spherical polar co-ordinates has a lowest energy eigenvalue $E_0$. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of $H$ satisfies

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0.$$

Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit on its value. **[7]**

Use a trial function of the form

$$F(\mathbf{r}, \alpha) = e^{-\alpha r/2}$$

where $\alpha$ is a variational parameter, to investigate the properties of a particle of mass $m$ in a central potential of the form

$$V(r) = V_0(r-3)e^{-r}$$

where $V_0$ is a positive constant.

Show that an upper bound on the ground state energy may be written

$$E(\alpha) = \frac{\hbar^2\alpha^2}{8m} - \frac{3V_0\alpha^4}{(1+\alpha)^4}$$

where $\alpha$ is a solution of the equation

$$\frac{\hbar^2(1+\alpha)^5}{4m} = 12V_0\alpha^2.$$ **[9]**

Determine the minimum value of $V_0$ such that the variational method predicts that there is just one bound state. **[4]**

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Note: If

$$I_n = \int_0^\infty e^{-ax} x^n dx$$

then for $n \geq 1$

$$I_n = \frac{n}{\alpha} I_{n-1}$$

and

$$\int_0^\infty e^{-ax} dx = \frac{1}{a}.$$

In spherical polar co-ordinates, $d\mathbf{r} = r^2 \sin\theta dr d\theta d\phi$, and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

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5. A charged spin-1/2 particle is fixed in an external uniform magnetic field $\mathbf{B} = B\hat{z}$ where $\hat{z}$ is a unit vector in the z-direction. The Hamiltonian operator is

$$H = \gamma B S_z$$

where $S_z$ is the z-component of the spin operator $\mathbf{S}$ and $\gamma$ is a constant.

Find the energy eigenvalues.

If $\alpha$ and $\beta$ are eigenvectors of $S_z$ corresponding to eigenvalues $\hbar/2$ and $-\hbar/2$ respectively,

determine $\omega$ such that the wave function at time $t$ is

$$\psi(t) = C_1 e^{-i\omega t/2}\alpha + C_2 e^{i\omega t/2}\beta$$

where $C_1$ and $C_2$ are constants. **[7]**

At time $t = 0$ the particle is in an eigenstate of $S_x$ corresponding to eigenvalue $\hbar/2$

$$\psi(0) = \frac{1}{\sqrt{2}}(\alpha + \beta)$$

Determine the times at which it will be in an eigenstate of $S_x$ corresponding to eigenvalue $-\hbar/2$ :

$$\psi(0) = \frac{1}{\sqrt{2}}(\alpha - \beta)$$ **[5]**

Find the expectation value of $S_x$ and of $S_x^2$ at time $t$. Hence calculate the uncertainty in $S_x$. Comment on the result. **[8]**

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$\mathbf{S} = \frac{\hbar}{2}\sigma$ where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ and in the basis formed by $\alpha$ and $\beta$ the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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**END OF PAPER**