UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

M.Sci.

Astronomy 2B11: Quantum Foundations of Astrophysics

COURSE CODE	:	**ASTR2B11**
UNIT VALUE	:	**0.50**
DATE	:	**10-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours 30 minutes**

02-C0077-3-30

TURN OVER

Answer SIX questions from section A and THREE questions from section B.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

SECTION A

[Part marks]

1. Explain what is meant by wave-particle duality, with reference to the work by de-Broglie and Young's two-slit experiment. [7]

2. State the **time-dependent** Schrödinger equation for a particle moving in a one-dimensional potential $V(x,t)$. [2]
   If the potential is not explicitly time-dependent, show that the wavefunction, at fixed energy E, takes the general form $u(x)\exp(-iEt/\hbar)$. [5]

3. Describe the tunnelling phenomenon of quantum mechanics. [3]
   Briefly discuss a physical process in which it is important. [4]

4. A particle in an infinite symmetrical well ($V = 0$ if $|x| < a$ and $V = \infty$ elsewhere) has wavefunction $u(x) = A \sin \frac{\pi x}{a}$. Calculate the expectation value of $\hat{P}_x$. [5]
   Sketch the probability of finding the particle at any point in the well. [2]

5. The operators corresponding to components of quantum angular momentum obey the relations $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. What is the physical significance of this result? [2]
   The ladder operators are defined by $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$. Evaluate $[\hat{L}_+, \hat{L}_-]$. [5]

6. What is meant by the statement that a wavefunction is normalised? [3]
   A particle is localised between $x = 0$ and $x = a$ and its wavefunction is $u(x) = Nx(a - x)$. Determine the normalisation constant N. [4]

7. A level of a given multi-electron atom is denoted by $^3D_1$. Explain the meaning of this symbol in full. [3]
   An excited state of helium has a configuration $2p3d$. Give all the corresponding terms in spectroscopic notation. [4]

8. Neglecting spin, the bound states of the hydrogen atom are described by the quantum numbers n, l, m. Explain their significance and give their possible values. [4]
What values of these quantum numbers are implied by the spectroscopic notation $5f$? [3]

SECTION B

9. Explain what is meant by the Heisenberg Uncertainty Principle for positions and momenta of a quantum particle. Discuss our current understanding of the Uncertainty Principle by considering in detail the following: [4]
(a) The Bohr microscope thought experiment
(b) The commutation relations $[\hat{P}_x, \hat{x}]$ and $[\hat{P}_y, \hat{x}]$ explaining their relevance to the Uncertainty Principle.
[16]

10. A one-dimensional potential step is defined by:

$$\text{Region 1} : x \leq 0 \ , V = 0$$
$$\text{Region 2} : x > 0 \ , V = V_0$$

A particle approaches the step with energy $E < V_0$ from region 1.

(a) Write down model solutions $u(x)$ to the Schrödinger equation in terms of reflected amplitudes R and transmitted amplitudes T as well as two wavenumbers k and q relevant to region 1 and 2 respectively, explaining your answers and defining the wavenumbers. [8]

(b) By matching wavefunctions and derivatives, obtain the transmitted and reflected amplitudes in terms of k and q. [8]

(c) If $|q| = 1/2$ evaluate the ratio $|u(0)|^2 : |u(1)|^2$ explaining the physical significance of this quantity. [4]

11. A particle moves in a one-dimensional potential well, with a potential $V = V_0$ for $|x| > a$ and $V = 0$ elsewhere. If $E < V_0$, the solution for the Time Independent Schrödinger equation (TISE) takes a different form in each of three spatial regions:
Region 1 : $u_1(x) = C\exp(-Px)$
Region 2 : $u_2(x) = D\exp(+Px)$
Region 3 : $u_3(x) = A\cos Kx + B\sin Kx$.
(a) Give the TISE for each of the three regions and identify the range of x corresponding to each of the three regions, justifying the form of the solutions. [10]

(b) Prove that, if $B = 0$, then:

$$\frac{P}{K} = \tan Ka$$

[6]

(c) In this case, what is the maximum number of quantum states with energy $E \leq (5^2\hbar^2\pi^2)/(8ma^2)$? [4]

12. The operator $\hat{L}_z$ can be expressed in terms of polar angles

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

(a) Derive the normalized eigenfunctions of $\hat{L}_z$, hence proving that they take the form $\frac{1}{\sqrt{2\pi}}e^{im\phi}$ and obtain the corresponding eigenvalues. [10]

(b) An electron is described by the following angular wavefunction:

$$u(\theta,\phi) = (\sqrt{\frac{1}{12\pi}} - 2i\sqrt{\frac{1}{8\pi}}\sin\theta\sin\phi).$$

Re-express u in terms of the spherical harmonics given below. Hence give the probability that a measurement will yield the eigenvalue of $\hat{L}^2$ equal to $2\hbar^2$. [6]

(c) What is the probability that $u(\theta,\phi)$ will yield the eigenvalue of $\hat{L}_z$ (i) equal to $-\hbar$? (ii) equal to zero? [4]

You may use the following:
$Y_{00}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}$
$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$
$Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$

13. The radial Schrödinger equation, in atomic units, for an electron in a hydrogen atom for which the orbital angular momentum quantum number, $\ell = 0$, is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} + 2E\right) F(r) = 0,$$

where E is the total energy.

(a) Put $F(r) = \exp(-r/\nu) y(r)$, where $E = -1/(2\nu^2)$, and show that

$$\frac{d^2 y}{dr^2} = \frac{2}{\nu}\left(\frac{d}{dr} - \frac{\nu}{r}\right) y.$$

[4]

(b) Assuming that $y(r)$ can be expanded as the series

$$y(r) = \sum_{p=0}^{\infty} a_p r^{p+1},$$

where $a_0 \neq 0$, show that the coefficients a_p in the series satisfy the recurrence relation,

$$p(p+1)a_p = \frac{2}{\nu}(p - \nu)a_{p-1}.$$

[8]

(c) Solutions of the radial Schrödinger equation exist which are bounded for all r provided that $\nu = n$, where n is a positive integer. Show that the un-normalized radial function for the $n = 2$ state is

$$F_{2s}(r) = a_0 r \left(1 - \frac{r}{2}\right) \mathrm{e}^{-\frac{r}{2}}.$$

[4]

(d) Show that the normalisation constant of the 2s state is $a_0 = \frac{1}{\sqrt{2}}$ and hence determine the expectation value of r^2 for this state. [4]

The following result may be assumed

$$\int_0^\infty r^m \mathrm{e}^{-\alpha r} dr = \frac{m!}{\alpha^{m+1}}.$$

END OF PAPER