Queen Mary
University of London

M.Sc. EXAMINATION

ASTMO41 Relativistic Astrophysics

20 May 2008 18:15-19:45
Duration: 1.5 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following:

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
Solar mass	$M_\odot$	2×10^{30} kg
Solar radius	$R_\odot$	7×10^5 km
1 pc		3.1×10^{16} m
1 AU		1.5×10^{11} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, j, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulae

The following results may be quoted without proof

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}.$$

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Gravitational radius of body of mass M: $r_g = 2GM/c^2 = 3(M/M_\odot)$ km.

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2}) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta) \sin^2\theta d\phi^2 +$$

$$\frac{2 r_g r a c}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, $a = \frac{J}{Mc}$ and J is angular momentum.

For the Schwarzschild and Kerr metric: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

The eikonal equation in a gravitational field:

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0,$$

where $\Psi = -\int k_i dx^i$, $k_i = g_{ik} k^i, k^i = \frac{dx^i}{d\lambda}$ and λ is an arbitrary scalar parameter.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R} \frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to the source and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) dM \text{ is the quadrupole tensor.}$$

SECTION A

Each question carries 20 marks. You should attempt ALL questions.

1. According to some spectroscopic measurements the redshift of spectral lines from a star of mass $m = 10M_\odot$ is $z = 4 \times 10^{-6}$.

(a) Derive expression for gravitational redshift in terms of the mass and radius of a gravitating body. Deduce the radius of the star, r.

[***14 Marks***]

(b) Estimate the ratio of the average density of the star to the density of the Sun.

[***6 Marks***]

2. Massive black hole is formed from a gas cloud of mass M and initial radius R_0.

(a) Given that at the moment of black hole formation the density of the cloud is 10^6 times larger than its initial density, show that

$$M \approx 5 \times 10^5 \left(\frac{R_0}{1\ \mathrm{AU}}\right) M_\odot.$$

[***12 Marks***]

(b) If $R_0 = 2$ AU, find the mass and density of black hole at the moment of its formation. Give the mass in solar masses and the density in kg m^{-3}.

[***8 Marks***]

3. (a) Explain why the surface where $g_{00} = 0$ is called the static limit and find the associated radial coordinate r_{SL} in the Kerr metric.

[***8 Marks***]

(b) Explain why the surface where $g^{11} = 0$ is called the event horizon and find the the associated radial coordinate r_{EH} in the Kerr metric.

[***12 Marks***]

SECTION B

Each question carries 40 marks. Only marks for the best ONE question will be counted.

1. Consider the propagation of a photon in the equatorial plane of a Schwarzschild black hole.

(a) Given that the solution of the eikonal equation can be written in the form

$$\Psi = -\omega t + \frac{b\omega}{c}\phi + \Phi_r(r),$$

where ω is the frequency of the photon and b is its impact parameter, show that

$$\frac{dr}{d\phi} = \pm\frac{r^2}{b}\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{r_g}{r}\right)}.$$

[***12 Marks***]

(b) Find all contravariant components of the wave vector for a photon,

$$k^i = \frac{dx^i}{d\lambda},$$

where λ is an arbitrary scalar parameter along the world-line. Demonstrate that λ does not appear in the final answer and present k^i in terms of r, ω and b.

[***12 Marks***]

(c) Show that photons can propagate along a circular orbit and find the radius of this orbit, r_{Ph}, and the corresponding value of the impact parameter b. Show that this orbit is unstable.

[***16 Marks***]

2. A supermassive black hole of mass M_{BH} is surrounded by a stellar cluster of radius R_{cl}, which consists of some hypothetical dark objects of mass m and radius r.

(a) Given that the average mass density within these objects is approximately equal to the density of the Sun, estimate the radius of tidal disruption, R_{TD}, in the gravitational field of the black hole and show that this radius does not depend on m and r.

[***9 Marks***]

 [This question continues overleaf . . .]

(b) Assume for simplicity that, as a result of tidal disruptions, material from disrupted stars forms a gas shell with inner radius $3r_g$ and outer radius R_{TD}. Assume also that the luminosity of the cluster, L_{cl}, is proportional to the volume of this gas shell. Show that the maximum value of L_{cl} is attained for $M_{BH} = M_{max} \approx 10^7 M_\odot$.

[*16 Marks*]

(c) Given that $R_{cl} = 100R_{TD}$ and $R_{TD} = 100r_g$, find M_{BH} and R_{cl}. Estimate the ratio L_{cl}/L_{max}, where L_{max} is the maximum possible luminosity of the cluster for a given radius.

[*15 Marks*]

3. (a) Consider a ring of test particles initially at rest in the (y, z)-plane, perturbed by a plane monochromatic gravitational wave propagating in the x-direction with frequency ω and amplitude h_0. Explain what is meant by "+" and "×" polarizations. Sketch the shape of the ring at times $t = 0, \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$ for two different polarizations of the gravitational wave: (i) $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$; and (ii) $h_+ = 0$, $h_\times = h_0 \sin\omega(t - x/c)$.

[*13 Marks*]

(b) A neutron star of mass m moves around a black hole of mass $M_{BH} \gg m$ on a circular orbit with radius r. The system emits gravitational radiation with amplitude h and frequency ω. Use the quadrupole formula to show that $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$ and T is the orbital period. Also show that to an order of magnitude

$$h \sim \frac{m}{M_{BH}} \frac{r_g^2}{rR},$$

where r_g is the gravitational radius of the black hole and R is the distance to the binary.

[*14 Marks*]

(c) If the binary is at the centre of the Andromeda galaxy, estimate M_{BH} if $\omega_0 = 2 \times 10^2$ Hz, $h_0 = 10^{-20}$ and $m = M_\odot$. You may assume that $R \approx 10^6$ pc.

[*13 Marks*]

[End of examination paper.]