QUEEN MARY,
UNIVERSITY OF LONDON
M.Sc. Astrophysics

ASTM003 Angular Momentum and Accretion in Astrophysics

28th May 2004 18:15 - 19:45

The duration of this examination is one and a half hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination.

The following notation is used throughout unless otherwise stated: The pressure, density, surface density, temperature, and magnetic field strength are denoted by P, ρ, Σ, T, $\mathbf{B}$ respectively. The effective temperature is denoted by T_{eff}, and opacity by κ. The mean molecular weight, gas constant, and kinematic viscosity are denoted by μ, $\mathcal{R}$, and ν, respectively. The permeability of free space is denoted by μ_0.

The gravitational constant $G = 7 \times 10^{-11}$ kg^{-1} m^3 s^{-2}. Stefan's constant $\sigma = 6 \times 10^{-8}$ J m^{-2} s^{-1} K^{-4}. The speed of light $c = 3 \times 10^8$ m s^{-1}. The solar mass $M_\odot = 2 \times 10^{30}$ kg. The solar radius $R_\odot = 7 \times 10^8$ m. The gas constant $\mathcal{R} = 8 \times 10^3$ J K^{-1} kg^{-1}.

The following mathematical identities may be useful:

The cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{i}}(A_j B_k - A_k B_j) - \hat{\mathbf{j}}(A_i B_k - A_k B_i) + \hat{\mathbf{k}}(A_i B_j - A_j B_i)$$

$\nabla \times \mathbf{A}$ in cylindrical polar coordinates:

$$\nabla \times \mathbf{A} = \hat{\mathbf{R}}\left(\frac{1}{R}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) - \hat{\phi}\left(\frac{\partial A_z}{\partial R} - \frac{\partial A_R}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{1}{R}\frac{\partial (R A_\phi)}{\partial R} - \frac{1}{R}\frac{\partial A_R}{\partial \phi}\right)$$

$\nabla.\mathbf{A}$ in cylindrical polar coordinates:

$$\nabla.\mathbf{A} = \frac{1}{R}\frac{\partial (R A_R)}{\partial R} + \frac{1}{R}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

SECTION A *You should attempt all questions. Marks awarded are shown next to the questions.*

1) [*15 marks*] The standard diffusion equation that describes the surface density evolution of a viscous accretion disc may be written:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial j}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3 \nu \Sigma \frac{d\Omega}{dR}\right)\right] \tag{1}$$

where j is the specific angular momentum, Ω is the angular velocity, Σ is the surface density, and ν is the kinematic viscosity.

Show that in a steady-state disc the following expression holds:

$$\nu\Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{2}$$

where $\dot{m}_d = -2\pi R \Sigma v_R$ is the constant rate of inwards mass flow, v_R is the radial velocity, and R_* denotes the position in the disc near to the central star where $d\Omega/dR = 0$. In deriving equation (2) you may use the expression

$$v_R \frac{\partial j}{\partial R} = \frac{1}{R\Sigma}\frac{\partial}{\partial R}\left[R^3 \nu \Sigma \frac{d\Omega}{dR}\right].$$

The region close to the star where the disc joins the stellar surface is known as the boundary layer. Explain briefly why this region is thought to provide a source of high energy photons that cannot be readily explained as arising from the main body of the disc. Provide one example of a scenario in which a boundary layer is thought to explain the emission of high energy photons.

2)[*15 marks*] Consider the growth of a large, gravitating, rocky planetary core by the accretion of smaller background solid objects during planet formation. Let the mass of the core be m_c, the core radius be R_c, the background object mass be m, and the number density of background objects be n. The accretion rate onto the core is given by

$$\frac{dm_c}{dt} = nmv\pi R_c^2 \left(1 + \frac{2Gm_c}{R_c v^2}\right) \tag{3}$$

where v is the velocity dispersion of the background objects. Using angular momentum and energy conservation, derive the gravitational focusing factor contained in brackets on the right hand side of equation (3).

Consider the situation where $Gm_c/R_c \gg v^2$. Show that the accretion rate by the planetary core in this case can be approximated by

$$\frac{dm_c}{dt} = \frac{2nm\pi G}{v} \left(\frac{3}{4\pi\rho_c}\right)^{1/3} m_c^{4/3}$$

where ρ_c is the density of the accreting core.

Show that the time required for the planetary core to grow from some initial mass m_{c0} to a very much larger mass is given by

$$t_g = \frac{3m_{c0}^{-1/3} v}{2\pi Gnm} \left(\frac{4\pi\rho_c}{3}\right)^{1/3}.$$

Explain briefly how estimates of the planetary growth time obtained from this last equation may be used to constrain theories of planet formation.

3) [*20 marks*] A rotating star has a strong, dipolar magnetic field that corotates with the star. The magnetic field threads through a surrounding circumstellar disc out to some radius. The interaction between the rotating magnetic field and the more slowly rotating parts of the disc cause these regions to be repelled from the star. This process may lead to the truncation of the disc inner edge at some distance, R_{in}, from the star.

The magnetic force per unit volume acting on the disc is given by

$$\mathbf{f} = \mathbf{j} \times \mathbf{B}$$

where $\mathbf{B}$ is the magnetic field strength and $\mathbf{j}$ is the current density. Ampére's law gives

$$\mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu_0}.$$

Working in cylindrical polar coordinates (R, ϕ, z), assuming that the system is axisymmetric, and that $B_R \ll B_z$ and $B_R \ll B_\phi$, show that torque per unit volume acting on the disc may be written

$$\mathcal{T} = \frac{R B_z}{\mu_0} \frac{\partial B_\phi}{\partial z}$$

where B_R, B_ϕ, and B_z are the R, ϕ, and z components of the magnetic field, respectively.

Hence show that the total torque becomes

$$\dot{J} = \int_{-H}^{H} \int_{R_{in}}^{\infty} \frac{2\pi R^2 B_z}{\mu_0} \frac{\partial B_\phi}{\partial z} dR dz$$

where H is the disc semi–thickness.

Assuming that the stellar magnetic field is such that

$$|\mathbf{B}(R)| = |\mathbf{B}_0| \left(\frac{R_*}{R} \right)^3,$$

where R_* is the stellar radius and $\mathbf{B}_0$ is a constant, and using simple arguments, show that an estimate of the torque is given by

$$\dot{J} = \frac{4\pi R_{in}^3}{3\mu_0} |\mathbf{B}_{in}|^2$$

where $\mathbf{B}_{in}$ is the magnetic field strength at the position R_{in}.

Question continues on next page.

Show further that the truncation radius of the disc, R_{in}, is given by

$$\frac{R_{in}}{R_*} = \left(\frac{4\pi |\mathbf{B_0}|^2 R_*^{5/2}}{3\sqrt{GM_*}\dot{m}_d \mu_0} \right)^{2/7}$$

where M_* is the mass of the star. In deriving this result you may assume that the viscous torque exerted at the inner edge of the disc is

$$\dot{J}_\nu = -3\pi\nu\Sigma R_{in}^2 \Omega(R_{in})$$

and $\dot{m}_d = 3\pi\nu\Sigma$. Here, ν is the kinematic viscosity, Σ is the disc surface density, and $\dot{m}_d$ is the steady state mass flow rate through the disc.

SECTION B *Each question carries 50 marks. You may attempt all questions but only marks for the best question will be counted.*

1) [*50 marks*] The virial theorem for a non–magnetised gaseous medium subject to internal pressure forces and self–gravity may be written

$$\frac{d^2 I}{dt^2} = 4\mathcal{K} + 2E_g + 6\int_V P dV \tag{4}$$

where the moment of inertia, I, and kinetic energy, $\mathcal{K}$, are given by

$$I = \int_V \rho r^2 dr, \qquad \mathcal{K} = \frac{1}{2}\int_V \rho v^2 dV.$$

Here r is the distance from the centre of mass (and origin of coordinate system), $\mathbf{v}$ is the velocity, P is the thermal pressure, and ρ is the density. E_g represents the gravitational potential energy which is given by:

$$E_g = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV dV'. \tag{5}$$

Derive equation (4), assuming only that the pressure goes to zero at the surface of the cloud. You may use the fact that the force per unit volume is given by

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P - \rho\nabla\Phi \tag{6}$$

where $\mathbf{v}$ is the velocity and the gravitational potential is given by

$$\Phi = -G\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'. \tag{7}$$

Consider an isothermal molecular cloud of mass M, rotating uniformly with angular velocity Ω, and within which the greatest distance between two points in the cloud is D. By assuming that the constituent gas obeys the ideal gas law, show that a sufficient condition for gravitational collapse of this cloud may be written as

$$\frac{GM}{D} > 2D^2\Omega^2 + \frac{6\mathcal{R}T}{\mu}.$$

Consider a molecular cloud of radius 3×10^{15} m and mass 4×10^{30} Kg uniformly rotating with angular velocity 10^{-14} radians s^{-1}. Estimate to order of magntitude the size of the protostellar disc that would result from the collapse of this cloud.

2) [*50 marks*] The radiative flux, F_ν, emitted at a particular frequency, ν, by a black–body with effective temperature T_{eff} is given by

$$F_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/KT_{eff}\right) - 1}, \tag{8}$$

where h is Planck's constant, K is Boltzmann's constant, c is the speed of light. Consider an axisymmetric accretion disc with a radial effective temperature profile given by

$$T_{eff} = \beta R^{-\alpha} \tag{9}$$

where α and β are positive constants. The effective temperature is T_{in} at the inner edge and T_{out} at the outer edge. Show that the luminosity, L_ν, at a particular frequency ν, is given by

$$\frac{1}{2}L_\nu = \left(\frac{2\pi}{c}\right)^2 \frac{K^{2/\alpha}}{\alpha}\nu^{3-2/\alpha}\beta^{2/\alpha}h^{1-2/\alpha}\int_0^\infty \frac{x^{2/\alpha-1}}{\exp\left(x\right) - 1}dx, \tag{10}$$

where $x = h\nu/(KT_{eff})$. In deriving this expression, you should only consider frequencies ν such that $KT_{in} \gg h\nu \gg KT_{out}$.

What does equation (10) tell you about the relation between the radial effective temperature distribution of an accretion disc and the emitted spectrum ?

A steady state accretion disc has $T_{eff} = \beta R^{-3/4}$. Make a sketch of $\log L_\nu$ versus $\log\nu$ for such a disc, and explain the shape of the curve. Your sketch should contain the regimes with $h\nu > KT_{in}$ and $h\nu < KT_{out}$.

3) [*50 marks*] Write an essay on *'Accretion Discs in Close Binary Systems'* giving both a qualitative and quantitative account of the physical processes involved. Use the bullet points below to guide the subject matter of your essay.

- Close binaries are classified according to whether their components are Roche–lobe filling or not. Describe this classification.
- Give the different subclasses of semi–detached systems, describing the nature of the accreting object and the donor star.
- Describe the Roche potential, and sketch the equipotential surfaces, labelling their important features.
- Explain how the onset of Roche–lobe overflow leads to the formation of a gaseous ring around the accreting star. Estimate the size of this ring for an equal mass binary system in terms of the binary separation, D. Explain how this gaseous ring spreads to become an accretion disc.
- Dwarf novae undergo recurrent outbursts. Sketch the lightcurve of such a system, indicating the typical duration of an outburst and the time between outbursts. Explain how the outburst duration can be used to constrain the magnitude of the viscosity operating in the accretion disc, and how this leads to the requirement for an anomalous source of viscosity.
- Discuss and explain the concept of Eddington limited accretion onto compact objects.
- Give estimates of the temperatures in the inner regions of accretion discs for the different classes of semi–detached binaries. Describe the region of the electromagnetic spectrum where the emission is expected to be observed in each case.