Queen Mary
University of London

M.Sc. EXAMINATION

ASTM001 Solar System Duration 3h

Thursday, 31 May 2007
18:15 – 21:15

This paper has two Sections, Section A and Section B:
you should attempt both Sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination.

Some useful numbers, definitions and identities:

- $G \approx 7 \times 10^{-11}$ m^3 Kg^{-1} s^{-2}
- $M_{\rm Earth} \approx 6 \times 10^{24}$ Kg
- $R_{\rm Earth} \approx 6 \times 10^{6}$ m
- $M_{\rm Sun} \approx 2 \times 10^{30}$ Kg
- $R_{\rm Sun} \approx 7 \times 10^{8}$ m
- MKS — Meters-Kilogrammes-Seconds
- $\mathrm{A} \times (\mathrm{B} \times \mathrm{C}) = \mathrm{B}\,(\mathrm{A} \cdot \mathrm{C}) - \mathrm{C}\,(\mathrm{A} \cdot \mathrm{B})$

SECTION A

Each question carries 5 marks (2.5 marks for each sub-part).
You should attempt <u>*ALL*</u> *five questions.*

1. Describe briefly (one or two sentences) what is meant by each of the following terms:
   (a) Pericentre
   (b) Mean anomaly

2. Give short (one or two sentences) answers to the following:
   (a) What does the Titius-Bode "Law" claim to explain? Give its quantitative relation (i.e., what is its "formula")?
   (b) Describe just one of its failures or non-physical features.

3. Describe briefly (one or two sentences) what is meant by each of the following terms:
   (a) Secular term in a perturbation solution
   (b) Tidal Love number

4. Describe briefly (one or two sentences) what is meant by each of the following terms:
   (a) Geostrophic balance
   (b) Cyclone

5. Describe briefly (one or two sentences) what is meant by each of the following terms:
   (a) Oligarchic growth
   (b) Olivine

[Next section overleaf.]

SECTION B

Each question carries 25 marks. There are 4 questions.
You may attempt all questions, but only marks for the best 3 questions will be counted.

1. Many instructive estimates can be made by assuming constant density and spherical symmetry for a Solar System body.

   (a) **[6 marks]** Derive the expression for the pressure at the centre, $p(0)$, of a uniform density spherical body of mass M and radius R that is in hydrostatic balance — namely,

   $$p(0) = \frac{3GM^2}{8\pi R^4}.$$

   (b) **[4 marks]** Using the expression in part (a), calculate $p(0)$ for the Earth. The actual estimated value, which does not assume a constant density, is $p_{\oplus}(0) = 3.6 \times 10^{12}$ Pa. Compare the two central pressures, $p(0)$ and $p_{\oplus}(0)$, and briefly comment on the implication of your calculation.

   (c) **[9 marks]** Still assuming constant density, show that the gravitational potential U of a spherical body is given by

   $$U = -\frac{3GM^2}{5R}.$$

   According to the virial theorem, a maximum of $-U/2$ can be radiated by the body over time. Given that the Sun's luminosity is $L_{\mathrm{Sun}} \approx 4 \times 10^{26}$ J s^{-1}, what is the cooling time implied by U and the constant density assumption? Comment on the your answer; does the answer suggest a source of internal heat other than a pure gravitational contraction, or collapse?

   (d) **[3 marks]** Rotation induces deviations from spherical symmetry. The deformation caused by rotation is characterised by Helmert's parameter μ_c, given by

   $$\mu_c = \frac{R^3\Omega^2}{GM},$$

   where Ω is the rotation rate of the body. Interpret this parameter physically; what force balance is being characterised?

   (e) **[3 marks]** Given an equatorial radius R_e and a polar radius R_p, the oblateness ϵ can be defined:

   $$\epsilon = \frac{R_e - R_p}{R}.$$

   The oblateness is also approximately equal to the Helmert's parameter – i.e., $\epsilon \approx \mu_c$. In the absence of any external torques, show by obtaining a relationship between ϵ and Ω that collapse (or compression) of a body enhances its oblateness.

2. Besides the conserved vector, angular momentum $\mathbf{L}$, the Kepler two-body problem possesses another conserved vector, known as the Laplace-Runge-Lenz vector $\mathbf{A}$.

(a) [4 marks] For a general central force $f(r)$, where r is the distance between the two bodies and $f < 0$, write down the equation of motion, relating the time derivative of momentum $\dot{\mathbf{p}}$ ($\equiv d\mathbf{p}/dt$) to $f(r)$, and $\mathbf{r}/r$ ($\equiv \hat{\mathbf{r}}$).

(b) [4 marks] By taking the cross product of $\dot{\mathbf{p}}$ with $\mathbf{L}$, show that

$$\dot{\mathbf{p}}\times\mathbf{L} = \frac{mf(r)}{r}\left[\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}}) - r^2\dot{\mathbf{r}}\right].$$

Then, show that

$$\frac{d}{dt}(\mathbf{p}\times\mathbf{L}) = -mf(r)\,r^2\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right).$$

(c) [4 marks] Assuming $f(r) = -k/r^2$, with k a positive constant, show that a conserved vector $\mathbf{A}$ exists, where

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\hat{\mathbf{r}}.$$

Then, describe the special direction to which $\mathbf{A}$ points. Describe the special plane in which $\mathbf{A}$ lies (fixed).

(d) [6 marks] Given that $\theta = \theta(t)$ is the angle between $\mathbf{A}$ and $\mathbf{r}$, show that

$$A\,r\cos\theta = l^2 - mkr,$$

where $A = |\mathbf{A}|$ and $l^2 = \mathbf{L}\cdot\mathbf{L}$. Hence, derive the orbit equation for the Kepler problem:

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right).$$

By comparing with the orbit equation obtained by solving the harmonic oscillator equation, evaluate A as a function of the orbital eccentricity e.

(e) [7 marks] Given that the eccentricity e is related to the total energy E by

$$e = \sqrt{1 + \frac{2El^2}{mk^2}},$$

obtain a relationship between A and l. We have identified three quantities $\{\mathbf{L}, \mathbf{A}, E\}$ as constants of the problem. According to this, how many conserved quantities are there in the problem? State how many of these are independent. Explain why.

[Next question overleaf.]

3. The *shallow-water equations* (SWE) for a thin layer of rotating fluid under gravity are

$$\begin{aligned} \frac{\partial \mathrm{v}}{\partial t} + \mathrm{v}\cdot\nabla\mathrm{v} &= -g\nabla h - \mathrm{f}\times\mathrm{v} \\ \frac{\partial h}{\partial t} + \mathrm{v}\cdot\nabla h &= -h\nabla\cdot\mathrm{v}, \end{aligned}$$

where $\mathrm{v} = (u, v)$ is the height-independent velocity in the thin layer [i.e., $u = u(x, y, t)$ and $v = v(x, y, t)$], $h = h(x, y, t)$ is the variable thickness of the layer with a constant average H, ∇ is the horizontal gradient, $\mathrm{f} = f\hat{\mathrm{e}}_z$ is the Coriolis parameter vector in the vertical direction with f a constant, and g is the gravity.

(a) **[4 marks]** The SWE admit gravity waves that propagate with phase speeds, $c_g = \sqrt{gH}$, when $f = 0$. Assuming that the average thickness of the atmosphere on the Earth ($\approx$10 Km) can be represented as H, estimate c_g. Compare this speed with typical sound and planetary (Rossby or "weather") wave speeds on the Earth.

(b) **[4 marks]** Consider atmospheric flow structures with characteristic size and speed of L and U, respectively. These characteristic scales lead to the nondimensional numbers, $F_r = U/\sqrt{gH}$ and $R_o = U/fL$. Physically interpret F_r and R_o. Given $f \approx 10^{-4}$ s^{-1} for the Earth, evaluate F_r and R_o for the weather scales on the Earth.

(c) **[6 marks]** The characteristic scales in part (b) give an associated timescale, $T = L/U$. By rescaling the velocity equation of the SWE, show that F_r and R_o are the natural parameters of the dynamics. In the case when F_r and R_o are both $\ll 1$, briefly comment on the physical situation suggested by the rescaled equation. In this situation, how must F_r and R_o be related in order for the equation to be "balanced"?

(d) **[6 marks]** For small amplitude disturbances, the SWE become

$$\begin{aligned} \frac{\partial \mathrm{v}}{\partial t} &= -g\nabla h - \mathrm{f}\times\mathrm{v} \\ \frac{\partial h}{\partial t} &= -H\nabla\cdot\mathrm{v}. \end{aligned}$$

Show that these equations in turn lead to the *Klein-Gordon equation* (KGE), which governs the linear adjustment of the atmosphere after perturbations, such as volcanic eruptions and asteroid impacts:

$$\frac{\partial^2 h}{\partial t^2} - c_g^2\nabla^2 h = -f^2 h.$$

Assume that the quantity, $\zeta - fh/H$, is zero initially with $\zeta = \hat{\mathrm{e}}_z\cdot\nabla\times\mathrm{v}$.

(e) **[5 marks]** From the KGE, obtain the following dispersion relation for Poincare (or inertia-gravity) waves:

$$\omega = \pm\sqrt{c_g^2 K^2 + f^2},$$

where $K^2 = k^2 + l^2$ with k and l the wavenumbers in $\hat{\mathrm{e}}_x$ and $\hat{\mathrm{e}}_y$ directions, respectively; ω is the wave frequency, giving the phase of the wave as $\{kx + ly - \omega t\}$.

[Next question overleaf.]

4. A small displacement, $\mathbf{s} = \mathbf{s}(\mathbf{r}, t)$, in a solid body is governed by the equation,

$$\frac{\partial^2 \mathbf{s}}{\partial t^2} = \frac{1}{\rho}(\lambda + \mu)\nabla\theta + \frac{\mu}{\rho}\nabla^2 \mathbf{s},$$

where $\mathbf{r}$ is the position of a body element, t is time, ρ is the density, λ and μ are constants related to the bulk modulus, and $\theta \equiv \nabla \cdot \mathbf{s}$ is a measure of the dilatation of the body element. Using the more convenient notation, $\partial_i \equiv \partial/\partial x_i = \nabla$ and $s_i \equiv \mathbf{s}$, we have $\theta = \partial_i s_i$. *(Note that here, and throughout this question, the summation convention is used.)*

(a) **[4 marks]** The quantity θ is related to the strain, or the symmetric stress, tensor: $e_{ij} \equiv \frac{1}{2}(\partial_i s_j + \partial_j s_i)$ — so-called because it is the symmetric part of the derivative tensor,

$$\partial_i s_j = \frac{1}{2}(\partial_i s_j + \partial_j s_i) + \frac{1}{2}(\partial_i s_j - \partial_j s_i).$$

Show that $\theta = e_{ii}$. If e_{ij} is the symmetric part of the derivative tensor, then $\frac{1}{2}(\partial_i s_j - \partial_j s_i)$ must be the antisymmetric part. Explain briefly what this antisymmetric part physically represents. Does it generate any stress?

(b) **[4 marks]** Solids behave elastically under small deformation, and we define Hooke's tensor C_{ijkh} such that $p_{ij} = C_{ijkh}\, e_{kh}$, where p_{ij} is the pressure tensor. In general, the energy density ω of the deformation is a quadratic form, $\omega = \frac{1}{2}C_{ijkh}\, e_{ij}e_{kh}$ (hence, $p_{ij} = \partial\omega/\partial e_{ij}$). However, using rotational symmetry, ω reduces to a simpler expression:

$$\omega = \frac{1}{2}\lambda\theta^2 + \mu\, e_{ij}e_{ij},$$

where λ and μ are the Lamé constants. What are the units (in MKS) of these constants? Show that p_{ij} is now given by $p_{ij} = \lambda\theta\delta_{ij} + 2\mu\, e_{ij}$, where δ_{ij} is the Kronecker delta.

(c) **[6 marks]** For expansions in three-dimensions, e_{ij} is given by $\frac{\theta}{3}\delta_{ij}$. Given this, show that in this case an 'effective pressure', $P_{\rm el} = -\mathcal{K}\theta$, is generated by the elasticity; here, $\mathcal{K}$ is the bulk modulus:

$$\mathcal{K} = \lambda + \frac{2}{3}\mu.$$

Expansion changes the density ρ according to $d\rho = -\rho\theta$, where $d\rho$ is a small change in density. Show that then the compressibility of a solid can be defined:

$$\frac{dp}{d\rho} = \frac{\mathcal{K}}{\rho},$$

where dp is a small pressure change induced by the elasticity.

(d) **[4 marks]** In the governing equation for $\mathbf{s}$, there is the factor '$\nabla^2\mathbf{s}$' in the last term on the right hand side. Show that $\nabla^2\mathbf{s} = \nabla\theta$ (i.e., $\partial_i^2 s_j = \partial_j\theta$) when the antisymmetric part of the derivative tensor vanishes. Then, derive the wave equation for the displacement vector $\mathbf{s}$.

(e) **[7 marks]** Derive the wave equations for the P-wave and the S-wave. Assume ρ is constant for each equation. What are the phase speeds of the P-wave and the S-wave? Given that $\mathcal{K} \approx 5\mu/3$ for the interior of the Earth, which one of the waves propagates faster in the interior, and by how much?

[End of examination paper.]