ASTM002 / MAS430 The Galaxy

May 2007 Examination Paper Model Answers

4th April, 2007, version

1. (a) Elliptical galaxies contain very little interstellar gas. There is a gradual trend in increasing gas fraction across spiral galaxy types from Sa to Sd and Sm. Irregulars are even more gas rich than late-type spirals. [Lectures] [2 marks]
Ellipticals have spectra composed of the integrated light of the constituent stars, and therefore show only absorption lines superimposed on the continuum. The stars are old and the continuum is strong in the red part of the spectrum. Irregular galaxies in contrast show strong emission lines from their interstellar gas superimposed on the stellar continuum. The gas will have participated in recent star formation and therefore the luminous young stars present produce a strong blue continuum. [Lectures] [1 mark]

(b) The total flux from an annulus of radius R and thickness $\mathrm{d}R$ is $2\pi R I(R)\mathrm{d}R$. Integrating this from the centre to arbitrary distance, the total flux is

$$F = \int_0^\infty 2\pi R\, I_0\, \mathrm{e}^{-(R/R_0)^{1/4}}\, \mathrm{d}R = 2\pi I_0 \int_0^\infty R\, \mathrm{e}^{-(R/R_0)^{1/4}}\, \mathrm{d}R = 2\pi I_0\, (4 \,.\, 7! R_0^2)$$

using the standard integral. Therefore, $F = 8!\pi I_0 R_0^2$.
[Full derivation unseen, but principles given in lecture] [3 marks]

(c) The Tully-Fisher relation provides a relation between luminosity L and rotational velocity v_{rot} of the form $L \propto v_{rot}^4$, for spiral galaxies. Observations of the rotation curve provide v_{rot}. The Tully-Fisher relation then provides L, which can be compared with the observed brightness to derive the luminosity distance. [Lectures] [2 marks]
The ratio of luminosities is $L_2/L_1 = (v_{rot\,2}/v_{rot\,1})^4 = (300/200)^4 = 5.1$. [Unseen] [2 marks]

(d) In dissipational collapse, mechanical (i.e. kinetic + potential energy) is dissipated (e.g. by heating gas). In dissipationless collapse, a system collapses but preserves the total mechanical energy. [Lectures] [2 marks]
Gas is collisional. If two gas clouds or gas flows meet, they will collide and dissipate energy. If the gas has a net angular momentum, the stable configuration will be a rotating disc. [Lectures] [2 marks]

(e) In the monolithic collapse model, the Galaxy formed from a large cloud of gas having initially a very low or zero metallicity. The cloud initially collapsed along near-radial paths, forming some stars in this time which in turn enriched the gas with heavy elements. Stars formation during this collapse produced the very low metallicity stars of the Galactic stellar halo, which have elongated orbits that are randomly oriented. These halo stars as a system have little net rotation. The angular momentum of the gas a short time later produced a rotating, flattened gas disc close to hydrostatic equilibrium. Stars that formed from the gas at this time produced the thick disc and were moderately metal-poor, relatively old, were rotationally supported but had some asymmetric drift and appreciable velocity dispersion. The gas settled to a thin disc, which formed stars to produce the stellar disc of the Galaxy. These stars have near-solar metallicity, near-zero asymmetric drift and small velocity dispersion. Gas that fell to the central regions produced the Bulge stars. [Lectures] [5 marks]

(f) Numerical simulations of structure and galaxy formation support the merger model. [Lectures] [1 marks]

[Total 20 marks for question]

2. (a) In the real galaxy, $T_{relax}/T_{cross} = 10^{11}/(12 \ln 10^{11}) \simeq 3 \times 10^6$.
In the N-body simulation using $\Phi = -Gm/r$ potential, $T_{relax}/T_{cross} = 10^5/(12 \ln 10^5) \simeq 7 \times 10^3$.
If the galaxy is $\simeq 10$ kpc $\simeq 3 \times 10^{20}$ m across with a typical velocity 300 kms^{-1} = 3×10^5 m s^{-1}, the crossing time will be $3 \times 10^{20}/3 \times 10^5$ s $= 10^{15}$ s $\simeq 3 \times 10^7$ yr. So the relaxation time in the model is $T_{relax} \simeq 7 \times 10^3 \times 3 \times 10^7$ yr $\simeq 2 \times 10^{10}$ yr. This is comparable to the age of the galaxy ($\simeq$ age of Universe). So relaxation will affect the simulation if the $-Gm/r$ potential is used, but not the real galaxy. [Lectures] [3 marks]

Two body gravitational interactions are suppressed in modelling by adding a softening parameter a to the potential. A potential $\Phi(r) = -Gm/\sqrt{r^2 + a^2}$ is adopted for the N-body modelling. [Lectures] [1 mark]

(b) Expressing the acceleration of a particle in terms of the gradient in the gravitational potential, and the rate of change of position as the velocity,

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial \Phi}{\partial x_i} \quad \text{and} \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i$$

for $i = 1$ to 3. But v_i and x_i are phase space coordinates, so v_i is independent of x_i. Φ is independent of velocity. Therefore,

$$\frac{\partial}{\partial x_i}\left(f \frac{\mathrm{d}x_i}{\mathrm{d}t}\right) = \frac{\partial}{\partial x_i}(f v_i) = v_i \frac{\partial f}{\partial x_i}$$

and

$$\frac{\partial}{\partial v_i}\left(f \frac{\mathrm{d}v_i}{\mathrm{d}t}\right) = -\frac{\partial}{\partial v_i}\left(f \frac{\partial \Phi}{\partial x_i}\right) = -\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}$$

So

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i}\right) = 0$$

which is the collisionless Boltzmann equation.
Alternatively, use Hamiltonian dynamics. [Lectures] [6]

(c) By substituting for the components of acceleration from $\mathrm{d}\mathbf{v}/\mathrm{d}t = -\boldsymbol{\nabla}\Phi$, we can express the collisionless Boltzmann equation as

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0\,,$$

To derive the first of the Jeans Equations, integrate this equation over all velocities at a point

$$\int \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) \mathrm{d}^3\mathbf{v} \;=\; \int 0 \,.\, \mathrm{d}^3\mathbf{v} \;.$$

$$\therefore \quad \int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} \;+\; \sum_{i=1}^{3} \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} \;-\; \sum_{i=1}^{3} \int \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} \;=\; 0 \;.$$

Because the integration is performed over all velocities for a given position and time,

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial}{\partial t} \int f \,\mathrm{d}^3\mathbf{v} \;=\; \frac{\partial n}{\partial t} \,,$$

using $n = \int f \,\mathrm{d}^3\mathbf{v}$. Similarly,

$$\int v_i \,\frac{\partial f}{\partial x_i} \,\mathrm{d}^3\mathbf{v} \;=\; \int \frac{\partial (v_i f)}{\partial x_i} \,\mathrm{d}^3\mathbf{v} \;=\; \frac{\partial}{\partial x_i} \int v_i \, f \, d^3\mathbf{v} \;=\; \frac{\partial \left(n \left\langle v_i \right\rangle \right)}{\partial x_i} \,,$$

on substituting $n\langle v_i \rangle = \int v_i \, f \,\mathrm{d}^3\mathbf{v}$. We also have

$$\int \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \,\mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \,\mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_i} (0) \;=\; 0 \,,$$

because $f \longrightarrow 0$ as $|v_i| \longrightarrow \infty$. Substituting for these terms,

$$\frac{\partial n}{\partial t} \;+\; \sum_{i=1}^{3} \frac{\partial n \langle v_i \rangle}{\partial x_i} \;=\; 0 \,,$$

the required result. [Lectures] [6]

(d) Multiplying out the expression for σ_{ij},

$$\begin{aligned}
\sigma_{ij}^2 \;&=\; \frac{1}{n} \int \left(v_i - \langle v_i \rangle \right) \left(v_j - \langle v_j \rangle \right) f \,\mathrm{d}^3\mathbf{v} \\
&=\; \frac{1}{n} \int \Big(\; v_i v_j - v_i \langle v_j \rangle - \langle v_i \rangle v_j + \langle v_i \rangle \, \langle v_j \rangle \; \Big) \; f \,\mathrm{d}^3\mathbf{v} \\
&=\; \frac{1}{n} \int v_i v_j \, f \,\mathrm{d}^3\mathbf{v} \;-\; \frac{1}{n} \int v_i \, \langle v_j \rangle \, f \,\mathrm{d}^3\mathbf{v} \;-\; \frac{1}{n} \int \langle v_i \rangle \, v_j \, f \,\mathrm{d}^3\mathbf{v} \\
&\qquad\quad +\, \frac{1}{n} \int \langle v_i \rangle \, \langle v_j \rangle \, f \,\mathrm{d}^3\mathbf{v} \\
&=\; \frac{1}{n} \int v_i v_j \, f \,\mathrm{d}^3\mathbf{v} \;-\; \langle v_j \rangle \frac{1}{n} \int v_i \, f \,\mathrm{d}^3\mathbf{v} - \; \langle v_i \rangle \frac{1}{n} \int v_j \, f \,\mathrm{d}^3\mathbf{v} \\
&\qquad\quad +\, \langle v_i \rangle \, \langle v_j \rangle \frac{1}{n} \int f \,\mathrm{d}^3\mathbf{v} \qquad \text{because } \langle v_i \rangle \text{ and } \langle v_j \rangle \text{ are constants} \\
&=\; \langle v_i v_j \rangle \;-\; \langle v_j \rangle \, \langle v_i \rangle \;-\; \langle v_i \rangle \, \langle v_j \rangle \;+\; \langle v_i \rangle \, \langle v_j \rangle
\end{aligned}$$

from the definition of the mean values. So,

$$\sigma_{ij}^2 \;=\; \langle v_i v_j\rangle \;-\; \langle v_i\rangle\,\langle v_j\rangle \quad .$$ [Lectures] [4]

[Total 20 marks for question]

3. (a) The velocity dispersion is isotropic and there is no net rotation. Therefore, $\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\phi^2\rangle = \sigma^2$, and σ is a constant here. From the Jeans equation given on the information page,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\sigma^2\right) \;+\; \frac{n}{r}\,(0) \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\;, \qquad \therefore\quad \sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\;,$$

on substituting for the velocity terms. Using $\Phi = v_0^2\ln r$ and $n(r) = kr^{-l}$ (where k is a constant) we get $\mathrm{d}\Phi/\mathrm{d}r = v_0^2/r$ and $\mathrm{d}n/\mathrm{d}r = -lkr^{-l-1}$. Substituting for these and cancelling r,

$$-\sigma^2\,lkr^{-l-1} \;=\; -kr^{-l}\,\frac{v_0^2}{r}\;, \qquad \therefore\quad \sigma \;=\; \frac{v_0}{\sqrt{l}}$$

For the Galactic halo, observations show that $n \propto r^{-3.5}$, i.e. $l = 3.5$. Using this and $v_0 = 220\ \mathrm{km\,s^{-1}}$, we get $\sigma \;\simeq\; 220\ \mathrm{km\,s^{-1}}/\sqrt{3.5} \;\simeq\; 120\ \mathrm{km\,s^{-1}}$. [Lectures] [4]
This is indeed what is observed. [Lectures] [1]

(b) The Galaxy is in a steady state, so $\partial(n\langle v_z\rangle)/\partial t = 0$, while from symmetry we expect $\partial(n\langle v_R v_z\rangle)/\partial R = 0$ and $n\langle v_R v_z\rangle/R = 0$. Therefore the equation in the question becomes

$$\frac{\partial(n\langle v_z^2\rangle)}{\partial z} \;=\; -\,n\,\frac{\partial\Phi}{\partial z}$$

[2]

If we consider only stars towards the Galactic poles, we consider only the z-direction, and Poisson's equation $\nabla^2\Phi = 4\pi G\rho$ reduces to

$$\frac{\partial^2\Phi}{\partial z^2} \;=\; 4\pi G\rho$$

(from the useful information at the start of the paper). So,

$$\frac{\partial}{\partial z}\left(-\frac{1}{n}\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\right) \;=\; 4\pi G\rho$$

[3]

Integrating perpendicular to the galactic plane from $-z$ to z, the surface mass density within a distance z of the plane at the solar Galactocentric radius R_0 is

$$\Sigma(R_0,z) \;=\; \int_{-z}^{z}\rho\,\mathrm{d}z' \;=\; \int_{-z}^{z}\frac{1}{4\pi G}\,\frac{\partial}{\partial z}\left(-\,\frac{1}{n}\,\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\right)\,\mathrm{d}z'$$

$$=\; -\,\frac{1}{4\pi G}\int_{z'=-z}^{z}\mathrm{d}\left(\frac{1}{n}\,\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\right) \;=\; -\,\frac{1}{2\pi G n}\,\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\bigg|_z$$

assuming symmetry about $z = 0$. [Lectures] [3]

(c) Imaging and spectroscopic observations can be made of stars towards the Galactic poles. The imaging data provide information about the star densities n as a function of height z from the Galactic plane. Distance information for this can come from photometric distance estimates. Line of sight velocities can be measured from spectroscopy, and these will be the same as the v_z velocity components because the stars lie towards the Galactic poles. The n and v_z data allow $\Sigma(R, z)$ to be determined as a function of z using the equation derived in question (c). This gives, after modelling the contribution from the dark matter halo, the mass density of the Galactic disc. [Lectures] [5]

(d) No significant dark matter is detected in the Galactic disc as a component of the disc itself (after subtracting the contribution from that part of the dark matter halo that extends through the disc). [Lectures] [2]

[Total 20 marks for question]

4. (a) Potential A: it is triaxial. The orbits are complex and fill a volume in space. [Lectures] [3]

Potential B: it is spherical. The orbit is confined to a plane and shows a classic rossette pattern in that plane. [Lectures] [3]

(b) At a distance r from the centre, the minimum speed v is 0. The maximum speed will be the escape velocity, corresponding to zero energy: $\frac{1}{2}mv^2 + m\Phi(r) = 0$. So, $\frac{1}{2}v^2 = -\Phi(r)$, and therefore $v = \sqrt{-2\Phi(r)}$. The limits on the integral over v are 0 to $\sqrt{-2\Phi(r)}$. [Lectures] [2]

The maximum value of the energy per unit mass is 0, corresponding to an unbound star (a star moving with the local escape velocity). The minimum value of E_m corresponds to zero kinetic energy at the point at radius r. Therefore, $E_m = \frac{1}{2}(0)^2 + \Phi(r) = \Phi(r)$. So the limits on the integral over E_m are $\Phi(r)$ (which is negative) to 0. [Lectures] [2]

(c) An integral of the motion is a parameter mesauring the dynamics of stars that is constant over stellar orbits. [Lectures] [2]

(d) Integrals of the motion: energy per unit mass; angular momentum per unit mass. [Principles stated in lectures] [1 mark each; total 2]

(e) The number density is related to the distribution function f by

$$n(r) = 4\pi \int_0^{\sqrt{-2\Phi(r)}} f\, v^2\, \mathrm{d}v$$

$$\therefore \quad n(r) = 4\pi \int_0^{\infty} \frac{n_0}{2\pi\sigma^2} \exp\left(-\frac{\frac{1}{2}v^2 + \Phi(r)}{\sigma^2}\right) v^2\, \mathrm{d}v \; ,$$

[Lectures] [1]

replacing the upper limit by ∞ because $f(v) \to 0$ only as $v \to \infty$ for this distribution function: this is not a stable potential over time. [Lectures] [1]

$$\therefore \quad n(r) = \frac{4\pi n_0}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{\Phi(r)}{\sigma^2}\right) \int_0^{\infty} v^2 \exp\left(-\frac{v^2}{2\sigma^2}\right) \mathrm{d}v \; .$$

We have $\int_0^\infty x^2 \mathrm{e}^{-ax^2} \mathrm{d}x = \frac{\sqrt{\pi/a^2}}{4}$. Put $a = 1/2\sigma^2$. This gives

$$\int_0^\infty x^2 \mathrm{e}^{-x^2/\sigma^2} \mathrm{d}x \;=\; \frac{\sqrt{\pi/(1/2\sigma^2)^3}}{4} \;=\; \frac{\sigma^3\sqrt{2\pi}}{2} \;.$$

Substituting for this,

$$n(r) \;=\; \frac{4\pi n_0}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{\Phi(r)}{\sigma^2}\right) \frac{\sigma^3\sqrt{2\pi}}{2} \;,$$

which gives the result

$$n(r) \;=\; n_0 \exp\left(-\frac{\Phi(r)}{\sigma^2}\right) \;.$$

[Lectures] [3]

This is consistent with the singular isothermal sphere

$$\rho(r) \;=\; \frac{\sigma^2}{2\pi G r^2}$$

[accept any sensible $\rho \propto 1/r^2$ or $n \propto 1/r^2$.] [Lectures] [1]

[Total 20 marks for question]

5. (a) A system of stars in a galaxy that is pressure supported has star orbits that predominantly randomly oriented. A system of stars that is rotationally supported has stars that predominantly orbit in similar directionsm and has a net angular momentum. [Lectures] [2]

(b) For a spherically symmetric profile, the continuity equation gives

$$\frac{\mathrm{d}M}{\mathrm{d}r} \;=\; 4\pi r^2 \rho$$

[1]

Integrating from the centre to a radius r,

$$\int_0^{M(r)} \mathrm{d}M \;=\; \int_0^r 4\pi\, r'^2 \rho(r')\, \mathrm{d}r'$$

$$M(r) - 0 \;=\; 4\pi \int_0^r r'^2 \frac{k}{(r'+a)(r'^2+a^2)} \mathrm{d}r'$$

$$\begin{aligned} M(r) \;&=\; 4\pi k \int_0^r \frac{r'^2}{(r'+a)(r'^2+a^2)} \mathrm{d}r' \\ &=\; 4\pi k \left[\frac{1}{2}\ln\left((r'+a)\sqrt{r'^2+a^2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{r'}{a}\right)\right]_0^r \\ &=\; 4\pi k \left(\frac{1}{2}\ln\left((r+a)\sqrt{r^2+a^2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{r}{a}\right) - \frac{1}{2}\ln\left(a\sqrt{a^2}\right) + \frac{1}{2}\tan^{-1}(0)\right) \end{aligned}$$

This gives

$$M(r) \;=\; 2\pi k \left(\ln \left(\left(1 + \frac{r}{a}\right) \sqrt{1 + \frac{r^2}{a^2}} \;-\; \tan^{-1}\left(\frac{r}{a}\right) \right) \right) \;,$$

the required result. [Unseen problem, but principles discussed in lectures.] [3]

The circular velocity v_{circ} is related to $M(r)$ by

$$v_{circ} \;=\; \sqrt{\frac{GM(r)}{r}} \;=\; \sqrt{\frac{2\pi Gk}{r} \left(\ln\left(1 + \frac{r}{a}\right) \sqrt{1 + \frac{r^2}{a^2}} \;-\; \tan^{-1}\left(\frac{r}{a}\right) \right)}$$

on substituting for $M(r)$. [Unseen application of principles from lectures.] [2]

For $r \gg a$, $\tan^{-1}(r/a) \sim \pi/2$. So $2\pi Gk/r$ dominates. So

$$v_{circ} \;\sim\; \frac{1}{\sqrt{r}}$$

[Unseen] [1]

(c) Poisson's equation gives $\nabla^2\Phi = 4\pi G\rho$. Therefore

$$\rho(r) \;=\; \frac{1}{4\pi G}\,\nabla^2\Phi \;.$$

The expression for $\nabla^2\Phi$ in the useful information at the start of the paper reduces to

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

in the case of spherical symmetry. [Lectures] [2]

Using $\Phi \;=\; -\,\dfrac{k}{r^2 + a^2}\;,\quad \dfrac{\mathrm{d}\Phi}{\mathrm{d}r} \;=\; +\,\dfrac{2kr}{(r^2 + a^2)^2}$ on differentiating. Therfore, $r^2\,\dfrac{\mathrm{d}\Phi}{\mathrm{d}r} \;=\; \dfrac{2kr^3}{(r^2 + a^2)^2}\;.$ Differentiating again,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \;=\; \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{2kr^3}{(r^2 + a^2)^2}\right) \;=\; \frac{6kr^3}{(r^2 + a^2)^2} - \frac{8kr^4}{(r^2 + a^2)^3} \;=\; \frac{2kr^2}{(r^2 + a^2)^3}\left(3a^2 - r^2\right)$$

Therefore the Laplacian is $\nabla^2\Phi \;=\; \dfrac{2k}{(r^2 + a^2)^3}\left(3a^2 - r^2\right)\;.$ This gives for the density,

$$\rho(r) \;=\; \frac{k}{2\pi G}\,\frac{(3a^2 - r^2)}{(r^2 + a^2)^3} \;,$$

the required result. [Unseen application of principles from lectures] [4]

The Jeans equation for spherical symmetry given at the front of the question paper gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\langle v_r^2\rangle\right) \;+\; \frac{n}{r}\left[\;2\langle v_r^2\rangle \;-\; \langle v_\theta^2\rangle \;-\; \langle v_\phi^2\rangle\;\right] \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;,$$

for a steady state system, using $\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\phi^2\rangle = \sigma^2$ for isotropy with no net rotation. Therefore,

$$\sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;,$$

[Lectures] [2]

because σ is constant over radius. Integrating from the centre to a radius r,

$$\int_{n_0}^{n(r)} \sigma^2 \frac{\mathrm{d}n}{n} = \int_{\Phi(0)}^{\Phi(r)} \mathrm{d}\Phi \ ,$$

where n_0 is the number density at the centre. Therefore,

$$\left[\sigma^2 \ \ln n\right]_{n_0}^{n(r)} = -\Phi(r) + \Phi(0) \ ,$$

because σ is constant. Therefore,

$$\sigma^2 \left(\ln n(r) - \ln n_0 \right) = \frac{k}{r^2 + a^2} - \frac{k}{a^2} \ ,$$

on substituting for $\Phi(r)$. This gives

$$\ln \left(\frac{n(r)}{n_0} \right) = - \frac{kr^2}{\sigma^2 a^2 (r^2 + a^2)} \ ,$$

and therefore the answer

$$n(r) = n_0 \exp \left(- \frac{kr^2}{\sigma^2 a^2 (r^2 + a^2)} \right) \ .$$

[Unseen] [3]

[Total 20 marks for question]

6. (a) CO is most easily observed at millimetre wavelengths (radio) [emitting at 1.3 and 2.6 mm]. This emission is produced by rotational transitions of the molecules. [Lectures] [2]

H_2 produces minimal emission and no radio lines. CO does emit readily at millimetre wavelengths and its distribution can be mapped directly. It is therefore used to trace the distribution of cold molecular gas. [Lectures] [1]

(b) One Balmer photon is produced by the interstellar gas for each Lyman continuum photon from the star [because a large majority of the H atoms in the gas are in the ground state]. Summing the Balmer emission, the Balmer luminosity of the H II region is 4.3×10^{48} photons s^{-1}. The star must therefore produce 4.3×10^{48} photons s^{-1} shortward of the Lyman limit. [Application of principles from lectures] [3]

The wavelength of 912 Å is that of the Lyman limit: photons shortwards of this ionise hydrogen atoms from the ground state. [Lectures] [1]

(c) Absorption lines from the interstellar gas can be observed when light from a bright background continuum source passes through the gas on its way to the observer. Single lines are often narrow. Complex molecular absorption is also observed, with a combination of rotational and vibrational transitions producing many lines and molecular bands. [Lectures] [2]

(d) Diffuse gas in the interstellar can be regarded as being part of one four (or three) distinct components: the cold neutral medium, the warm neutral medium, the warm ionised medium, and the hot ionised medium. These phases are pressure-confined and are stable in the long term. [Lectures] [3]

(e) The observed colour indices of an object that is affected by reddening by dust are (U – B) and (B – V). The intrinsic values (what would be observed if no dust were present) are $(\mathrm{U-B})_0$ and $(\mathrm{B-V})_0$. The colour excesses (which measure the extent of the reddening by dust) are therefore $E_{\mathrm{U-B}} = (\mathrm{U-B}) - (\mathrm{U-B})_0$ and $E_{\mathrm{B-V}} = (\mathrm{B-V}) - (\mathrm{B-V})_0$. So the observed values are $(\mathrm{U-B}) = (\mathrm{U-B})_0 + E_{\mathrm{U-B}}$ and $(\mathrm{B-V}) = (\mathrm{B-V})_0 + E_{\mathrm{B-V}}$. Substituting for these into the expression $Q \equiv (\mathrm{U-B}) - (E_{\mathrm{U-B}}/E_{\mathrm{B-V}})(\mathrm{B-V})$,

$$
\begin{aligned}
Q &= (\mathrm{U-B})_0 + E_{\mathrm{U-B}} - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}}\left((\mathrm{B-V})_0 + E_{\mathrm{B-V}}\right) \\
&= (\mathrm{U-B})_0 + E_{\mathrm{U-B}} - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}}(\mathrm{B-V})_0 + E_{\mathrm{U-B}} \\
&= (\mathrm{U-B})_0 - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}}(\mathrm{B-V})_0
\end{aligned}
$$

But $E_{\mathrm{U-B}}/E_{\mathrm{B-V}}$ is independent of extinction. So the Q parameter is equal to the value it would have in the absence of interstellar extinction. So the value of Q does not depend on the strength of the extinction. [Coursework problem] [2]

From the magnitudes, the observed colour indices are $(\mathrm{U-B}) = -0.43$ and $(\mathrm{B-V}) = 0.65$. This gives a Q parameter of $Q = -0.43 - 0.72(0.65) = -0.90$. Looking at the Q values in the table, the star is spectral type O5V and therefore has an intrinsic B–V colour index $(\mathrm{B-V})_0 = -0.35$. The colour excess is therefore $E_{\mathrm{B-V}} = (\mathrm{B-V}) - (\mathrm{B-V})_0 = 0.65 - (-0.35) = 1.00$. The V-band extinction is related to the B–V colour excess by $A_V \simeq 3.1 E_{\mathrm{B-V}}$. Therefore, $A_{\mathrm{V}} \simeq 3.1$ in this case. The intrinsic V-band magnitude is $\mathrm{V}_0 = \mathrm{V} - A_{\mathrm{V}} \simeq 11.00 - 3.1 \simeq 7.9$.

[Unseen, but similar to coursework problem] [3]

(f) The energy of the photon is $E_\gamma = hc/\lambda = 2.0 \times 10^-18$ J. The increase in the temperature of the grain is therefore $\Delta T = 2.0 \times 10^-18 / 2.0 \times 10^-21 = 1000$ K.

[Unseen application of principles described qualitatively in lectures] [2]

Before the ultraviolet photon is absorbed, the dust particle would radiate photons conforming to the Planck distribution for a temperature of 10 K: in the far infrared. After the heating, the particle would radiate in the near infrared. [This is the stochastic heating of dust grains.] [Described in lectures] [1]

[Total 20 marks for question]

7. (a) The [O/Fe] parameter is [O/Fe] = log(1/10) = −1.00.

[Application of definition from lectures] [1]

(b) Type II supernovae are produced by massive stars and eject enriched material into the interstellar medium $\sim 10^7$ yr after the initial star formation. This material is rich in C, N and O. In contrast, type Ia supernovae are probably caused by explosive fusion reactions in binary systems and eject enriched material $\gtrsim 10^8$ yr after the initial star formation. This material is rich in iron. Therefore, following an intial burst of star formation in low metallicity gas, the O/Fe ratio is high and Fe/H is low. As time passes, more iron is produced, the Fe/H ratio increases and O/Fe decreases. This can explain the trend in [O/Fe] against [Fe/H] in long-lived dwarf stars: [O/Fe] increases with decreasing [Fe/H] to [Fe/H] $\simeq -1$. [Lectures] [4]

(c) Some isotopes are built up by neutron capture. This process occurs slowly inside stars compared to the timescale for unstable nuclei to disintegrate through β decay, when it is called the s-process. Under extreme circumstances, such as in supernova explosions, neutron capture can occur rapidly compared to the β decay timescale and it is called the r-process. [Lectures] [2]

(d) The total mass in the volume is $M_{total} = M_{gas}(t) + M_{stars}(t)$, so a change δM_{gas} in the gas mass produces a change $\delta M_{stars} = -\delta M_{gas}$ in the mass in stars given that $M_{total} = M_{gas}(0)$ is constant. [1]
The change in metallicity is therefore

$$\delta Z \;=\; p\, \frac{\delta M_{stars}}{M_{gas}(0) - M_{stars}(t)} \quad .$$

[Lectures] [2]

Integrating from time 0 to t, the metallcity at time t is

$$Z(t) \;=\; p \int_0^{M_{stars}(t)} \frac{\mathrm{d}M_{stars}}{M_{gas}(0) - M_{stars}} \;=\; -\,p\,\ln\left(\frac{M_{gas}(0) - M_{stars}(t)}{M_{gas}(0)}\right)$$

which gives the required result. [Lectures] [2]

(e) The mean metallicity is $\langle Z\rangle \;=\; \frac{\int_0^{M_{\rm stars}} Z\,\mathrm{d}M'_{\rm stars}}{\int_0^{M_{\rm stars}} \mathrm{d}M'_{\rm stars}}$. Therefore,

$$\begin{aligned}
\langle Z\rangle \;&=\; \frac{1}{M_{\rm stars}} \int_0^{M_{\rm stars}} Z\,\mathrm{d}M'_{\rm stars} \;=\; \frac{1}{M_{\rm stars}} \int_0^{M_{\rm stars}} -\,p\ln\left(1 - \frac{M'_{\rm stars}}{M_{\rm gas}(0)}\right)\mathrm{d}M'_{\rm stars} \\
&=\; -\,\frac{p}{M_{\rm stars}}\left[\Big(M'_{\rm stars} - M_{\rm gas}(0)\Big)\ln\left(1 - \frac{M'_{\rm stars}}{M_{\rm gas}(0)}\right) \;-\; M'_{\rm stars}\right]_0^{M_{\rm stars}} \\
&=\; -\,\frac{p}{M_{\rm stars}}\left(\Big(M_{\rm stars} - M_{\rm gas}(0)\Big)\ln\left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)}\right) \;-\; M_{\rm stars} \;-\; 0\right) \\
&=\; p\left(\left(-1 + \frac{M_{\rm gas}(0)}{M_{\rm stars}}\right)\ln\left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)}\right) \;+\; 1\right)
\end{aligned}$$

which gives the required answer,

$$\langle Z\rangle \;=\; p\left(\frac{M_{\rm gas}(0)}{M_{\rm stars}} - 1\right)\ln\left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)}\right) \;+\; p\ .$$

[Coursework problem] [4]

Let the gas fraction be $\mu = M_{\rm gas}/M_{\rm gas}(0)$. But $M_{\rm stars} + M_{\rm gas} = M_{\rm total} = M_{\rm gas}(0)$, which gives, $M_{\rm gas} = M_{\rm gas}(0) - M_{\rm stars}$. Therefore, the gas fraction is

$$\mu \;=\; \frac{M_{\rm gas}(0) - M_{\rm stars}}{M_{\rm gas}(0)} \;=\; 1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)} \qquad \text{(A)}$$

and equivalently,

$$\frac{M_{\rm stars}}{M_{\rm gas}(0)} = 1 - \mu \qquad \therefore \qquad \frac{M_{\rm gas}(0)}{M_{\rm stars}} = \frac{1}{1-\mu} \qquad \text{(B)}$$

Substituting from (A) and (B),

$$\langle Z \rangle = p\left(\frac{1}{1-\mu} - 1\right)\ln\mu + p = p\left(\frac{1-(1-\mu)}{1-\mu}\right)\ln\mu + p = p\left(1 + \frac{\mu\ln\mu}{1-\mu}\right),$$

the required result. [Coursework problem] [1]

As gas is exhausted in star formation, $\mu \to 0$, $\mu\ln\mu/(1-\mu) \to 0$. Therefore, $\langle Z \rangle \to p$, the yield. [Coursework problem] [1]

(e) Globular clusters are metal-poor ([Fe/H] $\simeq -1$ to -2). So $N(\text{Fe})/N(H) \sim 1/10$th to $1/100$th solar.
Field halo stars are similarly metal-poor ([Fe/H] $\simeq -1.5$ to -2.
Thin disk stars have metallicities close to solar ([Fe/H] $\simeq +0.1$ to -0.3).
[Lectures] [2]

[Total 20 marks for question]

8. (a) The distance of a lens from the Galactic Centre is $r = \sqrt{D_L^2 + R_0^2}$ while the lens-source distance is $D_{LS} = R_0 - D_L$. The density at a lens at a distance D_L from the Earth is $\rho = \sigma^2/2\pi G(D_L^2 + R_0^2)$. Substituting for ρ into the expression for the optical depth given on the useful information page,

$$\begin{aligned}
\tau &= \frac{4\pi G}{c^2 D_S}\int_0^{D_S} D_L\, D_{LS}\,\rho(D_L)\,\mathrm{d}D_L = \frac{4\pi G}{c^2 R_0}\int_0^{R_0} \frac{D_L\,(R_0 - D_L)\,\sigma^2}{2\pi G(D_L^2 + R_0^2)}\,\mathrm{d}D_L \\
&= \frac{2\,\sigma^2}{c^2 R_0}\int_0^{R_0} \frac{D_L\,(R_0 - D_L)}{D_L^2 + R_0^2}\,\mathrm{d}D_L \\
&= \frac{2\,\sigma^2}{c^2 R_0}\left(R_0\int_0^{R_0} \frac{D_L}{D_L^2 + R_0^2}\,\mathrm{d}D_L - \int_0^{R_0} \frac{D_L^2}{D_L^2 + R_0^2}\,\mathrm{d}D_L\right) \\
&= \frac{2\,\sigma^2}{c^2 R_0}\left(R_0\left[\frac{1}{2}\ln\left(R_0^2 + D_L^2\right)\right]_0^{R_0} - \left[D_L - R_0\tan^{-1}\left(\frac{D_L}{R_0}\right)\right]_0^{R_0}\right) \\
&= \frac{2\,\sigma^2}{c^2 R_0}\left(R_0\left(\frac{1}{2}\ln\left(R_0^2 + R_0^2\right) - \frac{1}{2}\ln\left(R_0^2\right)\right) - R_0 + R_0\tan^{-1}\left(\frac{R_0}{R_0}\right)\right. \\
&\qquad\qquad \left. + 0 - R_0\tan^{-1}(0)\right) \\
&= \frac{2\,\sigma^2}{c^2 R_0}\left(\frac{R_0}{2}\ln(2R_0^2) - \frac{R_0}{2}\ln(R_0^2) - R_0 + R_0\tan^{-1}(1)\right) \\
&= \frac{\sigma^2}{c^2}\left(\ln 2 - 2 + \frac{\pi}{2}\right) = \frac{\sigma^2}{2c^2}\left(2\ln 2 + \pi - 4\right),
\end{aligned}$$

the required result.

[New application of principles discussed in lectures] [7]

Using $\sigma = 200\ \mathrm{km\,s^{-1}}$ and $c = 3.00 \times 10^8\ \mathrm{m\,s^{-1}}$, $\tau = 1.2 \times 10^{-7}$. If there are 5×10^5 stars, the probability of observing a microlensing event at any given time is about $1.2 \times 10^{-7} \times 5 \times 10^5 \simeq 6\,\%$. This gives a reasonable chance of observing a microlensing event if the globular cluster could be monitored for a few years (but the crowded star field could reduce success). [Unseen] [2]

(b) Typical optical depths for microlensing by hypothetical compact objects comprising the dark halo are very small ($\sim 10^{-7}$). Therefore very large numbers of stars $> 10^6$ must be monitored to give reasonable chances of detecting useful numbers of microlensing events, requiring fields with $> 10^6$ stars. These can be obtained in the Galactic Bulge, the Magellanic Clouds and some other Local Group galaxies. [Lectures] [2]

(c) Gravitational lensing preserves the colour and spectra of a source. A microlensing event will cause an amplification in apparent brightness that is the same in all photometric bands. This can exclude false detections caused by many types of variable star. [Lectures] [2]

(d) Differentiating $l = kt^n$, $\mathrm{d}l/\mathrm{d}t = nkt^{n-1}$, and again, $\mathrm{d}^2l/\mathrm{d}t^2 = n(n-1)kt^{n-2}$, Substituting for l and $\mathrm{d}^2l/\mathrm{d}t^2$ into the equation of motion,

$$n(n-1)\,k\,t^{n-2} = -\frac{GM}{k^2\,t^{2n}}$$

This requires the indices of t to be equal. So $n - 2 = -2n$, so $n = 2/3$. The coefficients must also be equal. Therefore,

$$n(n-1)k = -\frac{GM}{k^2}\,, \quad \text{and so,} \quad k^3 = -\frac{GM}{n(n-1)} = -\frac{GM}{\frac{2}{3}(\frac{2}{3}-1)} = \frac{9GM}{2}$$

This gives $k = \left(\frac{9GM}{2}\right)^{1/3}$. So the solution is $l = \left(\frac{9GM}{2}\right)^{1/3} t^{2/3}$.

[Generalised form of a coursework problem] [5]

The solution is inconsistent with the observed dynamics of M31 and the Galaxy. M31 and the Galaxy are observed to be approaching each other. This requires a solution in which the two galaxies initially moved apart, sharing the expansion of the Universe, but their mutual gravitation overcame this motion apart. They have since fallen towards each other.

The solution here corresponds to a limiting case where the mutual gravitation is just insufficient to halt the drift apart. The galaxies continue to move apart as time $t \to \infty$ but $\mathrm{d}l/\mathrm{d}t \to 0$. [Lectures] [2]

[Total 20 marks for question]