ASTM003 Angular Momentum and Accretion in Astrophysics
May 2006 Examination Paper Model Answers

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A1 (a) For circular motion, $GM/R^2 = \Omega^2 R$, at a radial distance R from a star of mass M, where $\Omega(R)$ is the angular velocity about the star, and G is the constant of gravitation.

$$\therefore \quad \Omega(R) = \sqrt{\frac{GM}{R^3}} = \sqrt{GM}\, R^{-\frac{3}{2}} \qquad \text{So} \quad \Omega(R) \propto R^{-\frac{3}{2}}\,.$$

[Lectures] [2 marks]

Differentiating, $\dfrac{d\Omega}{dR} = -\dfrac{3}{2}\sqrt{GM}\, R^{-\frac{5}{2}}$. Substituting for $\sqrt{GM} = \Omega\, R^{3/2}$,

$$\frac{d\Omega}{dR} = -\frac{3}{2}\frac{\Omega}{R}\,.$$

[Lectures] [2 marks]

The angular momentum of a mass m in a circular orbit of radius R is $J = mR^2\Omega$. The specific angular momentum is $j \equiv J/m = R^2\,\Omega = R^2\sqrt{GM/R^3} = \sqrt{GMR}\,.$

[Lectures] [1 mark]

(b) The gas at the edge of the cloud collapses to the edge of the disc, where it maintains a circular orbit of radius R_{max}. The specific angular momentum is $\sqrt{GMR_{max}} = j$ due to the conservation of angular momentum.

$$\therefore \quad R_{max} = \frac{j^2}{GM}$$

[Lectures] [3 marks]

(c) The minimum mass solar nebula is the protoplanetary gas/dust distribution that has the minimum mass required to form a planetary system analogous to the Solar System. [Lectures] [2 marks]

(d) The $\Sigma \propto R^{-3/2}$ relation is a power law approximation to the mass distribution of planets in the Solar System today. [Lectures] [2 marks]

The mass between radius R and $R + dR$ is $dM_D = 2\pi\, R\, dR\, \Sigma(R)$.

$$\therefore \quad \frac{dM_D}{dR} = 2\pi\, R\, \Sigma(R)$$

$$\int_0^{M_D(R)} dM'_D = \int_0^R 2\pi R'\, \Sigma(R')\, dR' = 2\pi \int_0^R R'\, k\, R'^{-3/2}\, dR'\ ,$$

on substituting $\Sigma(R) = kR^{-3/2}$.

$$\therefore \quad M_D(R) = 2\pi k \int_0^R R'^{-1/2}\, dR' = 2\pi k \left[2R'^{1/2}\right]_0^R = 4\pi k R^{1/2}$$

So $\quad M_D(R) \propto R^{1/2}$ [Lectures] [2 marks]

Substituting $k = \Sigma(R)\, R^{3/2}$, $M_D(R) = 4\pi\Sigma(R)\, R^{3/2}\, R^{1/2} = 4\pi\, R^2\, \Sigma(R)\,.$

$$\therefore \quad \Sigma(R) = \frac{M_D(R)}{4\pi R^2}$$

[Lectures] [1 mark]

[Total 15 marks for question]

A2 (a) Consider a planetesimal body of mass m moving with velocity v and impact parameter L towards the core. The impact parameter is so large that the body only just manages to be accreted: it is accreted at its closest approach to the core on the opposite side to its original direction of approach.

At closest approach its distance from the core centre is R_c, the radius of the core. Conservation of energy gives

Initial kinetic energy	=	Potential energy at closest approach	+	Kinetic energy at closest approach

$$\tfrac{1}{2}\,mv^2 \;=\; -\,\frac{Gm_c m}{R_c} \;+\; \tfrac{1}{2}\,mv_{app}^2$$

where v_{app} is the velocity relative to the core at closest approach. [Lectures] [4 marks]

$$\therefore \quad v_{app} \;=\; \sqrt{v^2 \;+\; \frac{2Gm_c}{R_c}} \;.$$

Conservation of angular momentum gives

angular momentum of planetesimal particle about centre of core initially	=	angular momentum of planetesimal particle about core centre at closest approach

$$\therefore \quad mLv \;=\; mR_c v_{app}$$

Substituting for v_{app},

$$L \;=\; R_c\sqrt{1 \;+\; \frac{2Gm_c}{R_c v^2}}$$

But L represents the effective radius for capturing planetesimal particles. So the cross section for capture is $A_{cap} = \pi L^2$.

$$\therefore \quad A_{cap} \;=\; \pi R_c^2\left(1 + \frac{2Gm_c}{R_c v^2}\right) \;,$$

the required result. [Lectures] [4 marks]

If the velocities of the planetesimals are v and their number density n, the number passing through area A_{cap} in unit time is nvA_{cap}. The mass capture rate is therefore

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} \;=\; nmv\pi L^2 \qquad \text{on substituting } A_{cap} = \pi L^2$$

$$\;=\; nmv\pi R_c^2\left(1 + \frac{2Gm_c}{R_c v^2}\right) \qquad \text{on substituting for } L$$

the required result. [Lectures] [4 marks]

When gravitational focusing dominates over the geometrical cross section,

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} \;\simeq\; nmv\pi R_c^2\left(\frac{2Gm_c}{R_c v^2}\right) \;=\; \frac{2\pi GnmR_c m_c v}{v^2}$$

Replacing the v in the numerator by $H_s\Omega$, and defining $\rho_s = nm$ to be the density of the layer of planetesimal bodies,

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} \simeq \frac{2\pi G \rho_s R_c m_c}{v^2} H_s\Omega \ ,$$

the required result. [Lectures] [3 marks]
We can approximate $\rho_s \simeq \Sigma_s/2H_s$. The density of the material in the planetesimals is $\rho_{gr} = \frac{m_c}{\frac{4}{3}\pi R_c^3}$, which gives the radius of the planetesimal as $R_c = \left(\frac{3m_c}{4\pi\rho_{gr}}\right)^{\frac{1}{3}}$. Substituting,

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} \simeq \frac{2\pi G \frac{\Sigma_s}{2H_s} \left(\frac{3m_c}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} m_c}{v^2} H_s\Omega \simeq \frac{\pi G \Sigma_s m_c^{\frac{4}{3}}}{v^2} \left(\frac{3}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} \Omega \ ,$$

the required result. [Lectures] [3 marks]
Integrating from time $t = 0$ to t_G,

$$\int_{m_c(0)}^{m_c(t_G)} m_c'^{-\frac{4}{3}} \,\mathrm{d}m_c' \simeq \int_0^{t_G} \frac{\pi G \Sigma_s}{v^2} \left(\frac{3}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} \Omega \,\mathrm{d}t$$

$$\therefore \quad \left[-3\, m_c'^{-\frac{1}{3}}\right]_{m_c(0)}^{m_c(t_G)} \simeq \frac{\pi G \Sigma_s}{v^2} \left(\frac{3}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} \Omega \left[\, t \,\right]_0^{t_G}$$

$$\therefore \quad -3\, m_c(t_G)^{-\frac{1}{3}} + 3\, m_c(0)^{-\frac{1}{3}} \simeq \frac{\pi G \Sigma_s}{v^2} \left(\frac{3}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} \Omega\, t_G$$

$$\text{Rearranging,} \qquad t_G \simeq \frac{3\left(m_c(0)^{-\frac{1}{3}} - m_c(t_G)^{-\frac{1}{3}}\right)}{\pi G \Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} \frac{v^2}{\Omega} \ ,$$

the required result. [Lectures] [5 marks]
But $m_c(t_G) \gg m_c(0)$. Therefore, $m_c(t_G)^{-\frac{1}{3}} \ll m_c(0)^{-\frac{1}{3}}$.

$$\therefore \quad m_c(0)^{-\frac{1}{3}} - m_c(t_G)^{-\frac{1}{3}} \simeq m_c(0)^{-\frac{1}{3}}$$

$$\text{So,} \qquad t_G \simeq \frac{3\, m_c(0)^{-\frac{1}{3}}}{\pi\, G\, \Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} \frac{v^2}{\Omega} \simeq \frac{3\, m_c(0)^{-\frac{1}{3}}}{\pi\, G\, \Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} H_s^2\, \Omega \ ,$$

on substituting $v \simeq H_s\Omega$. [Lectures] [2 marks]
[Total 25 marks for question]

A3 (a) Consider an element of gas of mass m. Its potential energy when at a large distance from the neutron star $\simeq 0$. Its potential energy when at the surface of the star is $-\,GMm/R$.
The potential energy released when it moves from a large distance to the surface of the neutron star is therefore $\simeq GMm/R$.
The potential energy released per unit mass of accreted gas falling to the surface

$$\simeq \frac{GM}{R} \simeq \frac{7\times10^{-11}\times3\times10^{30}}{10^4}\ \mathrm{J\,kg^{-1}} \simeq 21\times10^{15}\ \mathrm{J\,kg^{-1}} \simeq 2\times10^{16}\ \mathrm{J\,kg^{-1}}\,.$$

This energy is larger than the energy available through nuclear processes.

[Lectures] [4 marks]

For an element of gas of mass m,
potential energy when at large distance $\simeq 0$,
potential energy when in an orbit of radius $R_{in} = -\dfrac{GMm}{R_{in}}$.
Let v_{in} be the orbital velocity when in an orbit of radius R_{in}. For this orbit,

$$\frac{v_{in}^2}{R_{in}} = \frac{GM}{R_{in}^2} \qquad \therefore \quad v_{in}^2 = \frac{GM}{R_{in}}\,.$$

Kinetic energy of mass m is orbit of radius R_{in} is $\frac{1}{2}mv_{in}^2 = \frac{1}{2}m\frac{GM}{R_{in}} = \frac{GMm}{2R_{in}}$.

The total energy when in orbit around star $= \dfrac{GMm}{2R_{in}} - \dfrac{GMm}{R_{in}} = -\dfrac{GMm}{2R_{in}}$.

Therefore energy released in heating accretion disc

$= $ initial potential energy $-$ energy when in orbit $= 0 + \dfrac{GMm}{2R_{in}} = \dfrac{GMm}{2R_{in}}$.

So, kinetic energy when in orbit $=$ energy released in heating disc.
Therefore, energy released when falling on to star $=$ energy released in heating disc.

[Lectures, 3 marks]

A3 (b) In a steady state, the rate of energy released per unit area = rate of emission of electromagnetic radiation per unit area.

$$\therefore \quad \tfrac{9}{4}\,\Omega^2\,\nu\,\Sigma = 2\,\sigma\,T_{eff}^{\,4}$$

(the factor of 2 in the $2\sigma T_{eff}^4$ term accounts for the two sides of the accretion disc). Therefore,

$$T_{eff}^{\,4} = \frac{9}{8}\,\frac{\Omega^2\,\nu\,\Sigma}{\sigma} = \frac{9}{8}\,\frac{GM\,\nu\,\Sigma}{R^3\,\sigma} \quad \text{on subs.} \quad \Omega^2 = \frac{GM}{R^3}\,.$$

Therefore, $T_{eff} = \left(\dfrac{9}{8}\,\dfrac{GM\,\nu\,\Sigma}{R^3\,\sigma}\right)^{\frac{1}{4}}$, the required result. [Lectures] [3 marks]

[Total 10 marks for question]

B1 (a) Consider a ring in the accretion disc between radii R and $R+\mathrm{d}R$. The torque acting on the inner edge of this disc is

$$\mathcal{T}_{in} \;=\; -\,2\pi R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\;.$$

The torque acting on the outer edge is

$$\mathcal{T}_{out} \;=\; +\left[2\pi R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]_{R+\mathrm{d}R} \;=\; +\,2\pi R^3\,\nu\,\Sigma(R)\,\frac{\mathrm{d}\Omega}{\mathrm{d}R} \;+\; \frac{\partial}{\partial R}\left[2\pi R^3\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\,\mathrm{d}R$$

[Lectures] [4 marks]

The net torque acting on the ring is $\mathcal{T} = \mathcal{T}_{in} + \mathcal{T}_{out}$ [Lectures] [2 marks]

$$= \;-\,2\pi R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R} \;+\; 2\pi R^3\,\nu\,\Sigma(R)\,\frac{\mathrm{d}\Omega}{\mathrm{d}R} \;+\; \frac{\partial}{\partial R}\left[2\pi R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\,\mathrm{d}R$$

$$= \;\frac{\partial}{\partial R}\left[2\pi R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\,\mathrm{d}R$$

[Lectures] [3 marks]

The mass of the ring is $2\pi R\,\Sigma\,\mathrm{d}R$. The torque per unit mass on the ring is therefore

$$\mathcal{T}_m \;=\; \frac{1}{2\pi R\,\Sigma\,\mathrm{d}R}\,\frac{\partial}{\partial R}\left[2\pi R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\,\mathrm{d}R \;=\; \frac{1}{R\Sigma}\,\frac{\partial}{\partial R}\left[R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\;.$$

[Lectures] [3 marks]

The torque $\mathcal{T}$ drives a change in the angular momentum J through $\mathcal{T} = \dfrac{\mathrm{d}J}{\mathrm{d}t}$, where t is time. The torque per unit mass $\mathcal{T}_m$ is therefore related to the specific angular momentum j (the angular momentum per unit mass) by $\mathcal{T}_m = \dfrac{\mathrm{d}j}{\mathrm{d}t}$. But we have $\dfrac{\mathrm{d}j}{\mathrm{d}t} = v_R\dfrac{\partial j}{\partial R}$ (given in question), and also $j = R^2\,\Omega$. [Lectures] [4 marks]

$$\therefore \quad v_R\,\frac{\partial}{\partial R}\Big(R^2\Omega\Big) \;=\; \frac{1}{R\Sigma}\,\frac{\partial}{\partial R}\left[R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]$$

$$\therefore \quad R\,\Sigma\,v_R \;=\; \left(\frac{\partial}{\partial R}\Big(R^2\Omega\Big)\right)^{-1}\,\frac{\partial}{\partial R}\left[R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\;,$$

the required expression for $R\,\Sigma\,v_R$. [Lectures] [3 marks]

The continuity equation is $\dfrac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{.}(\rho\mathbf{v}) = 0$, and in cylindrical coordinates this becomes,

$$\frac{\partial\rho}{\partial t} \;+\; \frac{1}{R}\frac{\partial(R\rho v_R)}{\partial R} \;+\; \frac{1}{R}\frac{\partial(\rho v_\phi)}{\partial\phi} \;+\; \frac{\partial(\rho v_z)}{\partial z} \;=\; 0\;.$$

[Lectures] [4 marks]

In this axisymmetric system we have $\partial/\partial\phi = 0$ and $\partial/\partial z = 0$.

$$\therefore \quad \frac{\partial\rho}{\partial t} \;+\; \frac{1}{R}\frac{\partial(R\rho v_R)}{\partial R} \;=\; 0\;.$$

[Lectures] [4 marks]

Integrating over z, $$\int_{-\infty}^{\infty} \frac{\partial \rho}{\partial t}\,\mathrm{d}z \;+\; \int_{-\infty}^{\infty} \frac{1}{R}\,\frac{\partial (R\rho v_R)}{\partial R}\,\mathrm{d}z \;=\; 0$$

$$\therefore \quad \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \rho\,\mathrm{d}z \;+\; \frac{1}{R}\,\frac{\partial}{\partial R}\left(R\int_{-\infty}^{\infty}\rho\,\mathrm{d}z\; v_R\right) \;=\; 0$$

But $\Sigma \equiv \int_{-\infty}^{\infty} \rho\,\mathrm{d}z$. $\quad \therefore \quad \dfrac{\partial \Sigma}{\partial t} + \dfrac{1}{R}\,\dfrac{\partial}{\partial R}\,(R\,\Sigma\,v_R) \;=\; 0$. [Lectures] [5 marks]

Substituting for $R\,\Sigma\,v_R$,

$$\frac{\partial \Sigma}{\partial t} \;+\; \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial(R^2\Omega)}{\partial R}\right)^{-1} \frac{\partial}{\partial R}\left(R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\right] \;=\; 0 \;,$$

the required result. [2 marks]

For a Keplerian disc, $\Omega(R) = \sqrt{GM/R^3}$, where M is the mass of the central star and G is the constant of gravitation.

$$\therefore \quad \frac{\mathrm{d}\Omega}{\mathrm{d}R} \;=\; -\,\frac{3}{2}\sqrt{\frac{GM}{R^5}}$$

$$\text{and} \qquad \frac{\partial}{\partial R}(R^2\Omega) \;=\; \frac{\partial}{\partial R}\left(R^2\sqrt{\frac{GM}{R^3}}\right) \;=\; \frac{\partial}{\partial R}\left(\sqrt{GM}\;R^{\frac{1}{2}}\right) \;=\; \frac{1}{2}\,\frac{\sqrt{GM}}{\sqrt{R}}$$

[Lectures] [3 marks]

Substituting for these,

$$\frac{\partial \Sigma}{\partial t} \;+\; \frac{1}{R}\,\frac{\partial}{\partial R}\left[\left(\frac{1}{2}\,\frac{\sqrt{GM}}{\sqrt{R}}\right)^{-1} \frac{\partial}{\partial R}\left(-\,R^3\,\nu\,\Sigma\,\frac{3}{2}\sqrt{\frac{GM}{R^5}}\right)\right] \;=\; 0$$

$$\therefore \quad \frac{\partial \Sigma}{\partial t} \;+\; \frac{1}{R}\,\frac{\partial}{\partial R}\left[\frac{2\sqrt{R}}{\sqrt{GM}}\,\frac{\partial}{\partial R}\left(-\,R^{\frac{1}{2}}\,\nu\,\Sigma\,\frac{3}{2}\sqrt{GM}\right)\right] \;=\; 0$$

$$\therefore \quad \frac{\partial \Sigma}{\partial t} \;-\; 3\,\frac{1}{R}\,\frac{\partial}{\partial R}\left[R^{\frac{1}{2}}\,\frac{\partial}{\partial R}\left(R^{\frac{1}{2}}\,\nu\,\Sigma\right)\right] \;=\; 0$$

$$\therefore \quad \frac{\partial \Sigma}{\partial t} \;=\; \frac{3}{R}\,\frac{\partial}{\partial R}\left[R^{\frac{1}{2}}\,\frac{\partial}{\partial R}\left(R^{\frac{1}{2}}\,\nu\,\Sigma\right)\right] \;,$$

the required result. [Lectures] [3 marks]

(b) The equation for $\partial\Sigma/\partial t$ in part (a) involves only surface density Σ, time t, radius R, kinematic viscosity ν and dimensionless numerical constants. So we expect the evolutionary timescale τ_{ev} to depend only on Σ, R and ν.
The dimensions of τ_{ev} are [T], those of Σ are $[\mathrm{M}]\,[\mathrm{L}]^{-2}$, those of R are [L], and those of ν are $[\mathrm{L}]^2\,[\mathrm{T}]^{-1}$.
Assume that $\tau_{ev} = k\,\Sigma^a\,R^b\,\nu^c$, where a, b, c and k are dimensionless numerical constants. From a dimensional analysis,

$$[\mathrm{T}] \;=\; [\mathrm{M}]^a\,[\mathrm{L}]^{-2a}\,[\mathrm{L}]^b\,[\mathrm{L}]^{2c}\,[\mathrm{T}]^{-c} \;.$$

Equating the dimensions of time, $1 = -c \quad \therefore \quad c = -1$.
Of mass, $a = 0$.
Of length, $0 = -2a + b + 2c$. Using $a = 0$ and $c = -1$, $b = 2$.
Therefore, $\tau_{ev} = k\,\Sigma^0\,R^2\,\nu^{-1}$. Hence,

$$\tau_{ev} \;=\; k\,\frac{R^2}{\nu}\;,$$

the required result. [5 marks]

(c) The sound speed c_s, the half-thickness H and the angular velocity Ω are related by $c_s = \Omega H$ at a distance R from the star (a result of hydrostatic equilibrium). Substituting for c_s into $\nu = \alpha c_s H$, we get $\nu = \alpha H^2 \Omega$. But for Keplerian rotation, $\Omega = \sqrt{GM/R^3}$. Therefore,

$$\nu \;=\; \alpha H^2 \sqrt{GM}\, R^{-3/2} \;=\; \alpha \sqrt{GM}\left(\frac{H}{R}\right)^2 R^{1/2}\;,$$

the required result. [Unseen] [5 marks]
[Total 50 marks for question]

B2 The volume element $\mathrm{d}V$ can be represented as a mass element $\mathrm{d}m$ with $\mathrm{d}m = \rho\,\mathrm{d}V$.

$$I \;=\; \int_V \rho\, r^2\,\mathrm{d}V \;=\; \int_V r^2\,\mathrm{d}m \;=\; \int_V \mathbf{r}\,.\,\mathbf{r}\,\mathrm{d}m\;.$$

Differentiating with respect to time t,

$$\frac{\mathrm{d}I}{\mathrm{d}t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{r}\,.\,\mathbf{r}\,\mathrm{d}m \;=\; \int_V \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\,.\,\mathbf{r})\;\mathrm{d}m \;=\; \int_V \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,.\,\mathbf{r} \;+\; \mathbf{r}\,.\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)\mathrm{d}m$$

$$= \;2\int_V \mathbf{r}\,.\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}m$$

[Lectures] [4 marks]

Differentiating again,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; 2\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{r}\,.\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}m \;=\; 2\int_V \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}\,.\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)\mathrm{d}m$$

$$= \;2\int_V \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,.\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \;+\; \mathbf{r}\,.\,\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)\mathrm{d}m \;=\; 2\int_V \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^2 \mathrm{d}m \;+\; 2\int_V \mathbf{r}\,.\,\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\,\mathrm{d}m$$

[Lectures] [3 marks]

The total kinetic energy is $\mathcal{K} \;=\; \int_V \tfrac{1}{2}\,\mathrm{d}m \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^2 \;=\; \tfrac{1}{2}\int_V \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^2 \mathrm{d}m$

So $\quad 2\int_V \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^2 \mathrm{d}m \;=\; 4\,\mathcal{K}\;.$ [Lectures] [2 marks]

From Newton's second law on an element of mass $\mathrm{d}m$,

$$\mathrm{d}m\,\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \;=\; \mathrm{d}\mathbf{F} \qquad \text{where } \mathrm{d}\mathbf{F} \text{ is the force on the mass element}$$

$$=\; \mathbf{f}\,\mathrm{d}V \qquad \text{where } \mathbf{f} \text{ is the force per unit volume}$$

$$\therefore \quad \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 4\mathcal{K} + 2\int_V \mathbf{r}\cdot\mathbf{f}\,\mathrm{d}V \ ,$$

the required result. [Lectures] [3 marks]

The pressure contribution to $2\int_V \mathbf{r}\cdot\mathbf{f}\,\mathrm{d}V$ is

$$\begin{aligned}
2\int_V \mathbf{r}\cdot(-\boldsymbol{\nabla}P)\,\mathrm{d}V &= -2\int_V (\mathbf{r}\cdot\boldsymbol{\nabla})P\,\mathrm{d}V \\
&= -2\int_V \Big(\boldsymbol{\nabla}\cdot(P\,\mathbf{r}) - P\,\boldsymbol{\nabla}\cdot\mathbf{r}\Big)\,\mathrm{d}V \qquad \text{using the identity for } (\mathbf{A}\cdot\boldsymbol{\nabla})f \\
&= -2\int_V \Big(\boldsymbol{\nabla}\cdot(P\,\mathbf{r}) - 3P\Big)\,\mathrm{d}V \qquad \text{using } \boldsymbol{\nabla}\cdot\mathbf{r} = 3 \\
&= -2\int_V \boldsymbol{\nabla}\cdot(P\,\mathbf{r})\,\mathrm{d}V + 6\int_V P\,\mathrm{d}V
\end{aligned}$$

[Lectures] [7 marks]

But $\int_V \boldsymbol{\nabla}\cdot(P\,\mathbf{r})\,\mathrm{d}V = \int_S (P\,\mathbf{r})\cdot\mathrm{d}\mathbf{S}$ using Gauss's theorem

$= 0$ because $P = 0$ on the surface S.

Therefore the pressure contribution to $2\int_V \mathbf{r}\cdot\mathbf{f}\,\mathrm{d}V = 6\int_V P\,\mathrm{d}V$.

[Lectures] [5 marks]

The acceleration due to gravity of an element of the fluid is $\mathbf{g} = -\boldsymbol{\nabla}\Phi$. Therefore the gravitational force per unit volume is $\mathbf{f} = \rho\,\mathbf{g} = -\rho\,\boldsymbol{\nabla}\Phi$. The gravitational contribution to $2\int_V \mathbf{r}\cdot\mathbf{f}\,\mathrm{d}V$ is therefore

$$2\int_V \mathbf{r}\cdot(\rho\,\mathbf{g})\,\mathrm{d}V = -2\int_V \rho(\mathbf{r})\,\mathbf{r}\cdot\boldsymbol{\nabla}\Phi\,\mathrm{d}V = 2E_g \ ,$$

where E_g is the internal gravitational potential energy using the result in the question. [4 marks]

The final virial theorem result for the non-magnetised cloud is therefore

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 4\mathcal{K} + 2E_g + 6\int_V P\,\mathrm{d}V$$

[Lectures] [3 marks]

The ideal gas law gives the pressure $P = \dfrac{\mathcal{R}}{\mu}\,\rho\,T$. The gas contribution to $\mathrm{d}^2 I/\mathrm{d}t^2$ in the isothermal cloud is

$$6\int_V P\,\mathrm{d}V = 6\int_V \frac{\mathcal{R}}{\mu}\,\rho\,T\,\mathrm{d}V = \frac{6\mathcal{R}T}{\mu}\int_V \rho\,\mathrm{d}V$$

because the temperature is constant

$$= \frac{6\mathcal{R}}{\mu}\,T\,M \qquad \text{because } \int \rho\,\mathrm{d}V = M\text{, the total mass.}$$

[4 marks]

The condition for the collapse of the cloud is $\dfrac{\mathrm{d}^2 I}{\mathrm{d}t^2} < 0$. [Lectures] [3 marks]

The initial kinetic energy is $\mathcal{K} = 0$ (the cloud is stationary). Substituting for $\mathcal{K}$ and E_g,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 0 + 2\left(-\frac{3}{5}\,\frac{GM^2}{R}\right) + \frac{6\mathcal{R}}{\mu}\,TM = -\frac{6}{5}\,\frac{GM^2}{R} + \frac{6\mathcal{R}}{\mu}\,TM$$

For collapse to occur,

$$-\frac{6}{5}\frac{GM^2}{R} + \frac{6\mathcal{R}}{\mu}TM < 0$$

which on rearranging gives, $M > \dfrac{5\mathcal{R}T}{\mu G}R$, the required result.

[New application of principles from lectures] [6 marks]

Using $T = 10\,\mathrm{K}$, $R = 3\times10^{15}$ m and $\mu = 2\,\mathrm{g\,mol^{-1}} = 0.002\,\mathrm{kg\,mol^{-1}}$,

$$M > \frac{5\times8.3\times10\times3\times10^{15}}{0.002\times6.6\times10^{-11}}\ \mathrm{kg} \sim 10^{31}\ \mathrm{kg} \sim \frac{10^{31}}{2.0\times10^{30}}M_\odot \sim 5\,M_\odot$$

So $M \gtrsim 5\,M_\odot$. The minimum mass is 10^{31} kg $\sim 5\,M_\odot$. This is a stellar mass, so stellar mass clouds can collapse to form protostars. [Unseen] [6 marks]

[Total 50 marks for question]

B3 The energy emitted at frequency ν per unit surface area per unit time per unit frequency interval is F_ν. Integrating over the disc, the power radiated from one face of the disc is

$$\tfrac{1}{2}L_\nu = \int_{R_{in}}^{R_{out}} 2\pi R\,F_\nu\,\mathrm{d}R\ ,$$

where R_{in} and R_{out} are the inner and outer radii of the accretion disc.

[Lectures] [2 marks]

Substituting for F_ν,

$$\begin{aligned}\tfrac{1}{2}L_\nu &= \int_{R_{in}}^{R_{out}} 2\pi R\,\frac{2\pi h\nu^3}{c^2}\,\frac{\mathrm{d}R}{\exp\left(h\nu/kT_{eff}\right)-1}\\ &= \left(\frac{2\pi}{c}\right)^2 h\int_{R_{in}}^{R_{out}}\frac{\nu^3\,R\ \ \mathrm{d}R}{\exp\left(h\nu/kT_{eff}\right)-1}\end{aligned}$$

[Lectures] [4 marks]

Use the substitution $x \equiv \dfrac{h\nu}{kT_{eff}}$. (Here $T_{eff} = T_{eff}(R)$ and $x = x(R)$.) [4 marks]

$$\therefore\ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{h\,\nu}{k\,T_{eff}}\right)\mathrm{d}R = \frac{\mathrm{d}}{\mathrm{d}T_{eff}}\left(\frac{h\,\nu}{k\,T_{eff}}\right)\frac{\mathrm{d}T_{eff}}{\mathrm{d}R}\,\mathrm{d}R = -\frac{h\,\nu}{k\,T_{eff}^2}\frac{\mathrm{d}T_{eff}}{\mathrm{d}R}\,\mathrm{d}R$$

$$\therefore\ \mathrm{d}R = -\frac{k\,T_{eff}^2}{h\,\nu}\frac{\mathrm{d}R}{\mathrm{d}T_{eff}}\,\mathrm{d}x$$

[4 marks]

Let $x = x_{in}$ when $R = R_{in}$ (and $T_{eff} = T_{in}$)
and $x = x_{out}$ when $R = R_{out}$ (and $T_{eff} = T_{out}$)
Substituting,

$$\begin{aligned}\tfrac{1}{2}L_\nu &= \left(\frac{2\pi}{c}\right)^2 h\int_{x_{in}}^{x_{out}}\frac{\nu^3\,R}{\mathrm{e}^x-1}\left(-\frac{k\,T_{eff}^2}{h\,\nu}\right)\frac{\mathrm{d}R}{\mathrm{d}T_{eff}}\,\mathrm{d}x\\ &= \left(\frac{2\pi}{c}\right)^2\int_{x_{in}}^{x_{out}} -\,k\,T_{eff}^2\,\frac{\nu^2\,R}{\mathrm{e}^x-1}\,\frac{\mathrm{d}R}{\mathrm{d}T_{eff}}\,\mathrm{d}x\end{aligned}$$

[4 marks]

But $T_{eff} \;=\; \beta\, R^{-\alpha}$. $\quad\therefore\quad R \;=\; \beta^{\,1/\alpha}\, T_{eff}^{\,-1/\alpha}$

Differentiating, $\quad \dfrac{\mathrm{d}R}{\mathrm{d}T_{eff}} \;=\; -\,\dfrac{1}{\alpha}\,\beta^{\frac{1}{\alpha}}\, T_{eff}^{\,-\frac{1}{\alpha}-1}$

Substituting for $\mathrm{d}R/\mathrm{d}T_{eff}$ and for $R \;=\; \beta^{1/\alpha}\, T_{eff}^{-1/\alpha}$,

$$\begin{aligned}
\tfrac{1}{2}\,L_\nu &= \left(\frac{2\pi}{c}\right)^2 \int_{x_{in}}^{x_{out}} -\,k\,T_{eff}^2\,\frac{\nu^2}{\mathrm{e}^x-1}\,\left(\beta^{1/\alpha}\,T_{eff}^{\,-1/\alpha}\right)\left(-\,\frac{1}{\alpha}\,\beta^{\frac{1}{\alpha}}\,T_{eff}^{\,-\frac{1}{\alpha}-1}\right)\,\mathrm{d}x \\
&= \left(\frac{2\pi}{c}\right)^2 \int_{x_{in}}^{x_{out}} \frac{k}{\alpha}\,\frac{\nu^2}{\mathrm{e}^x-1}\,\beta^{\frac{2}{\alpha}}\,T_{eff}^{\,1-\frac{2}{\alpha}}\;\mathrm{d}x
\end{aligned}$$

Substituting for $T_{eff} = h\nu/kx$,

$$\begin{aligned}
\tfrac{1}{2}\,L_\nu &= \left(\frac{2\pi}{c}\right)^2 \int_{x_{in}}^{x_{out}} \frac{k}{\alpha}\,\frac{\nu^2}{\mathrm{e}^x-1}\,\beta^{\frac{2}{\alpha}}\left(\frac{h\nu}{kx}\right)^{1-\frac{2}{\alpha}}\;\mathrm{d}x \\
&= \left(\frac{2\pi}{c}\right)^2 \int_{x_{in}}^{x_{out}} \frac{k^{\frac{2}{\alpha}}}{\alpha}\,\frac{\nu^{3-\frac{2}{\alpha}}}{\mathrm{e}^x-1}\,\beta^{\frac{2}{\alpha}}\,x^{\,\frac{2}{\alpha}-1}\,h^{\,1-\frac{2}{\alpha}}\;\mathrm{d}x \\
&= \left(\frac{2\pi}{c}\right)^2 \frac{k^{\frac{2}{\alpha}}}{\alpha}\,\nu^{3-\frac{2}{\alpha}}\,\beta^{\frac{2}{\alpha}}\,h^{\,1-\frac{2}{\alpha}} \int_{x_{in}}^{x_{out}} \frac{x^{\,\frac{2}{\alpha}-1}}{\mathrm{e}^x-1}\;\mathrm{d}x
\end{aligned}$$

[8 marks]

If we avoid extremes in frequency (i.e. we consider frequencies ν such that $kT_{in} \gg h\nu \gg kT_{out}$), we can approximate $x_{in} \simeq 0$, $x_{out} \simeq \infty$.

$$\tfrac{1}{2}\,L_\nu \;=\; \left(\frac{2\pi}{c}\right)^2 \frac{k^{\frac{2}{\alpha}}}{\alpha}\,\nu^{3-\frac{2}{\alpha}}\,\beta^{\frac{2}{\alpha}}\,h^{\,1-\frac{2}{\alpha}} \int_0^\infty \frac{x^{\,\frac{2}{\alpha}-1}}{\mathrm{e}^x-1}\;\mathrm{d}x\;\;,$$

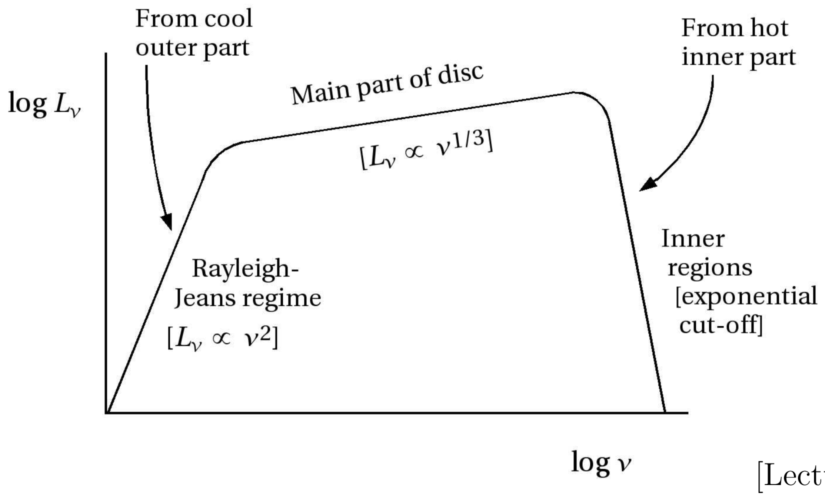
the required result.

[Application of method in lectures to a general power law model] [6 marks]

The relationship between L_ν and ν is a power law: $L_\nu \propto \nu^{3-\frac{2}{\alpha}}$. [Unseen]

$$T_{eff} \;=\; \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\right]^{\frac{1}{4}} \;=\; \left[\frac{3GM\dot{m}}{8\pi\sigma}\right]^{\frac{1}{4}}\; R^{-\frac{3}{4}}\;.$$

This is of the form $T_{eff} \;=\; \beta\, R^{-\alpha}$ where $\beta \;=\; \left[\dfrac{3GM\dot{m}}{8\pi\sigma}\right]^{\frac{1}{4}}$ and $\alpha = 3/4$. This gives $L_\nu \,\propto\, \nu^{3-\frac{2}{\alpha}} \,\propto\, \nu^{3-\frac{8}{3}} \,\propto\, \nu^{\frac{1}{3}}$. So the spectrum has the form $L_\nu \,\propto\, \nu^{\frac{1}{3}}$ except at the extremes of the range in frequency. [6 marks]



[Lectures] [7 marks]

The luminosity depends on the mass flow rate $\dot{m}$ through β, with $L_\nu \propto \beta^{\frac{2}{\alpha}}$, $\alpha = 3/4$ and $\beta = \left[\frac{3GM\dot{m}}{8\pi\sigma}\right]^{\frac{1}{4}}$. So, $L_\nu \propto \left(\left[\frac{3GM\dot{m}}{8\pi\sigma}\right]^{\frac{1}{4}}\right)^{\frac{8}{3}}$ $\therefore$ $L_\nu \propto \dot{m}^{\frac{2}{3}}$

There is therefore a power-law dependence on the mass inflow rate with index 2/3.

[Unseen] [5 marks]

Total 50 marks for question]