

Thermal & Kinetic Physics 2011

Exam Solutions

**Thermal & Kinetic Physics (PHY-214) Exam 2011**  
**Solutions.**

**SECTION A. Attempt answers to all questions**

**Question A1**

**Solution**

$$dU = TdS - PdV$$

$dU$  is the incremental change in internal energy.

$dS$  is the incremental change in entropy.

$-PdV$  is the incremental work done on the system where  $P$  is the pressure and  $dV$  the incremental change in volume.

**Question A2**

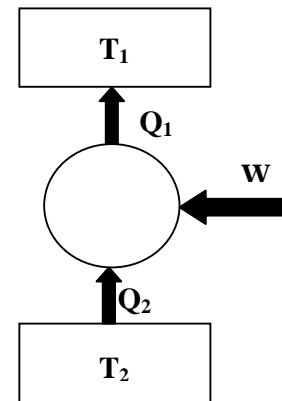
**Solution**

For isolated systems  $\Delta S \geq 0$  where the equality refers to reversible changes in the thermodynamic equilibrium state.

**Question A3**

**Solution**

$$\eta_R = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{T_2}{T_1 - T_2}$$



**Question A4**

**Solution**

$$S = k_B \ln \Omega$$

$k_B$  is the Boltzmann constant

**Question A5**

**Solution**

$$\eta_{HP} = \frac{Q_1}{W} = \frac{Q_1}{Q_1 - Q_2}$$

## Thermal & Kinetic Physics 2011

### Exam Solutions

#### Question A6

##### Solution

$$P_i V_i^\gamma = P_f V_f^\gamma \quad \text{and} \quad P = \frac{RT}{V}$$

Substituting to replace  $P$  with  $T$

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

#### Question A7

##### Solution

$$C_P = \left( \frac{\partial H}{\partial T} \right)_P$$

$H = U + PV$  is the enthalpy ( $U$  is the internal energy and  $P$  and  $V$  the pressure and volume of the gas).

Or 
$$C_P = T \left( \frac{\partial S}{\partial T} \right)_P$$

$S$  is the entropy and  $T$  the temperature of the gas.

#### Question A8

##### Solution

$$H = U + PV = \frac{7}{2}PV + PV = \frac{9}{2}nRT = \frac{9}{2}Nk_B T$$

$$\langle KE \rangle = \frac{3}{2}k_B T$$

#### Question A9

##### Solution

$$\kappa = -V \left( \frac{\partial P}{\partial V} \right)_T$$

#### Question A10

##### Solution

$$W = - \int_{V_i}^{V_f} P dV = -P[2V - V] = -PV$$

$P$  and  $V$  are both positive quantities and therefore  $W$  is negative meaning the work is done by the gas on its surroundings.

# Thermal & Kinetic Physics 2011

## Exam Solutions

### SECTION B. Attempt two of the four questions in this section

B1) a)

i) Solution

$$dS = \frac{\delta Q_R}{T}$$

Where  $dS$  is the incremental change in the entropy of a system as heat  $\delta Q_R$  flows reversibly at temperature  $T$ . If the heat flows from the system then following sign convention it is negative and the entropy change is negative.

ii) Solution

$$\oint_{\text{System cycle}} \frac{dQ(T)}{T} \leq 0$$

b)

i) Solution

There is no net heat flow as the process takes place in an adiabatic container so we calculate the final temperature

$$\Delta Q_{H_2O} = 6\text{kg} \times C_p \times (T_f - 298.15\text{K})$$

$$\Delta Q_{ice} = 0.5\text{kg} \times C_p \times (T_f - 273.15\text{K}) + 0.5\text{kg} \times \ell^{\text{melting}}$$

And

$$\Delta Q_{ice} + \Delta Q_{H_2O} = 0.5\text{kg} \times C_p \times (T_f - 273.15\text{K}) + 0.5\text{kg} \times \ell^{\text{melting}} + 6\text{kg} \times C_p \times (T_f - 298.15\text{K}) = 0$$

$$6.5\text{kg} \times C_p \times T_f = C_p (273.15 \times 0.5\text{kg} + 298.15 \times 6\text{kg}) - 0.5\text{kg} \times \ell^{\text{melting}}$$

$$T_f = \frac{(273.15 \times 0.5\text{kg} + 298.15 \times 6\text{kg})}{6.5\text{kg}} - \frac{0.5\text{kg} \ell^{\text{melting}}}{6.5\text{kg} C_p}$$

$$T_f = \frac{1.925 \times 10^3 \text{kgK}}{6.5\text{kg}} - \frac{0.5\text{kg} \cdot 3.33 \times 10^5 \text{Jkg}^{-1}}{6.5\text{kg} \cdot 4.2 \times 10^3 \text{Jkg}^{-1}\text{K}^{-1}} = 2.901 \times 10^2$$

## Thermal & Kinetic Physics 2011

### Exam Solutions

$$T_f = 290.1 \text{ K} = 16.95 \text{ }^\circ\text{C}$$

*The final state of the system is 6.5 kg of water at 290.1 K and at 1 atmosphere.*

#### ii) Solution

*This calculation is done in three parts;*

a) *The net entropy change for the water initially at 298.15 K cooling to 290.1 K is*

$$\Delta S_{H_2O} = 6\text{kg} \times 4.2 \times 10^3 \text{ JK}^{-1}\text{kg}^{-1} \int_{298.15}^{290.1} \frac{dT}{T} = 2.52 \times 10^4 \ln \frac{290.1}{298.15} = -689.75 \text{ JK}^{-1}$$

b) *The entropy change of ice in the phase change ice  $\rightarrow$  water is*

$$\Delta S_{Ice \rightarrow H_2O} = \frac{0.5\text{kg} \times 3.33 \times 10^5}{273.15} = +610.1 \text{ JK}^{-1}$$

c) *The entropy change as the ice-water warms from 273.15 K to 290.1 K is*

$$\Delta S_{Ice-water} = 0.5\text{kg} \times 4.2 \times 10^3 \text{ JK}^{-1}\text{kg}^{-1} \int_{273.15}^{290.1} \frac{dT}{T}$$

$$\Delta S_{Ice-water} = \left( 2.1 \times 10^3 \ln \frac{290.1}{273.15} \right) \text{ JK}^{-1} = +126.43 \text{ JK}^{-1}$$

$$\Delta S_{Ice} = 610.1 + 126.43 \text{ JK}^{-1} = +736.53 \text{ JK}^{-1}$$

$$\Delta S_{Net} = \Delta S_{Ice} + \Delta S_{H_2O} = (736.53 - 689.75) \text{ JK}^{-1} = +46.78 \text{ JK}^{-1}$$

#### iii) Solution

*The mixing process is not reversible as without further input 6.5kg of water at 290.1 K will not spontaneously become 6kg of water at 298.15 K plus 0.5kg of ice.*

#### iv) Solution

*As the process is irreversible we expect from the Second Law that  $\Delta S > 0$  and this is seen to be the case in part ii)*

c)

#### i) Solution

## Thermal & Kinetic Physics 2011

### Exam Solutions

*The final temperature of the mix can be found by making the total heat flow zero*

$$mC_P[(T_f - T_1) + (T_f - T_2)] = 0$$
$$T_f = \frac{T_1 + T_2}{2}$$

*The entropy change of the universe is simply the entropy changes of the two masses of water combined;*

$$\Delta S = mc_P \left[ \int_{T_1}^{T_f} \frac{dT}{T} + \int_{T_2}^{T_f} \frac{dT}{T} \right] = mc_P \left[ \ln\left(\frac{T_f}{T_1}\right) + \ln\left(\frac{T_f}{T_2}\right) \right]$$
$$\Delta S = mc_P \left[ \ln\left(\frac{T_f^2}{T_1 T_2}\right) \right] = mc_P 2 \ln\left(\frac{(T_1 + T_2)/2}{\sqrt{T_1 T_2}}\right) \quad \underline{\text{QED}}$$

#### ii) Solution

*For the Second Law to be obeyed requires that  $\Delta S > 0$  as we have an irreversible process. This requires that the argument of the logarithm is greater than 1. I.e. that*

$$\frac{T_1 + T_2}{2\sqrt{T_1 T_2}} > 1$$

#### iii) Solution

*To demonstrate that the second Law is obeyed it is necessary to demonstrate the inequality of part ii) is generally true for all  $T_1$  and  $T_2$  as follows.*

$$T_1 + T_2 > 2\sqrt{T_1 T_2}$$

*Squaring both sides of the inequality*

$$(T_1 + T_2)^2 > 4T_1 T_2$$

$$T_1^2 + T_2^2 + 2T_1 T_2 > 4T_1 T_2$$

$$T_1^2 + T_2^2 - 2T_1 T_2 = (T_1 - T_2)^2 > 0$$

*This is always true if  $T_1$  and  $T_2$  are real numbers and therefore the Second Law is obeyed.*

## Thermal & Kinetic Physics 2011

### Exam Solutions

B2) a)

i) Solution

$$F = U - TS$$

$$dF = dU - TdS - SdT$$

*From the first law*  $dU - TdS = -PdV$

*and we simplify dG*

$$dF = -SdT - PdV$$

*We can see that the natural variables of F are T and V allowing us to write*

$$F = F(T, V)$$

ii) Solution

*We use the natural variables of G from part i) to write the incremental dG as*

$$dF = \left( \frac{\partial F}{\partial T} \right)_V dT + \left( \frac{\partial F}{\partial V} \right)_T dV$$

*By comparing the coefficients of the increments of the natural variables in the two equations we find*

$$S = - \left( \frac{\partial F}{\partial T} \right)_V \qquad P = - \left( \frac{\partial F}{\partial V} \right)_T$$

iii) Solution

*By making use of the fact that dF is a perfect differential we may write,*

## Thermal & Kinetic Physics 2011

### Exam Solutions

$$\frac{\partial^2 F}{\partial T \partial V} = \frac{\partial^2 F}{\partial V \partial T} \quad \Rightarrow \quad \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial V} \right) = \frac{\partial}{\partial V} \left( \frac{\partial F}{\partial T} \right)$$

*This allows us to write*

$$\left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial P}{\partial T} \right)_V$$

*Which is the required Maxwell relation.*

**b)**

**i) Solution**

*The heat capacity at constant volume is defined as*

$$C_V = \text{Limit}_{\Delta T \rightarrow 0} \frac{\Delta Q}{\Delta T}$$

*At constant volume the first law becomes  $\Delta U = \Delta Q - PdV = \Delta Q$*

*And  $C_V$  may thus be given in terms of partial differential of a state function as*

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V$$

*Using the ideal gas equation of state for a monatomic gas,  $U = \frac{3}{2}PV = \frac{3}{2}nRT$*

*From the expression for  $C_V$  we have*

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V = \frac{3}{2}nR$$

**ii) Solution**

*The thermodynamic identity for a P-V-T system is;*

$$dU = TdS - PdV$$

*dividing the thermodynamic identity by  $dV$  at constant  $T$*

## Thermal & Kinetic Physics 2011

### Exam Solutions

$$\left. \frac{dU}{dV} \right|_{T \text{ const}} = T \left. \frac{dS}{dV} \right|_{T \text{ const}} - P \left. \frac{dV}{dV} \right|_{T \text{ const}}$$

Which may be written as

$$\left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial S}{\partial V} \right)_T - P$$

### iii) Solution

From part ii) we have

$$\left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial S}{\partial V} \right)_T - P$$

From part i) we found

$$\left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial P}{\partial T} \right)_V$$

$$\left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial P}{\partial T} \right)_V - P$$

Rearrange vdW equation of state

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}$$

$$\left( \frac{\partial P}{\partial T} \right)_V = \frac{nR}{V - nb}$$

Therefore

$$\left( \frac{\partial U}{\partial V} \right)_T = T \frac{nR}{V - nb} - P = \frac{nRT}{V - nb} - \frac{nRT}{V - nb} + \frac{n^2 a}{V^2} = \frac{n^2 a}{V^2}$$

**QED**

### iv) Solution

$$\left( \frac{\partial U}{\partial T} \right)_V = C_V = \frac{3}{2} nR$$



# Thermal & Kinetic Physics 2011

## Exam Solutions

And from part iii)

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{n^2 a}{V^2}$$

Integrating these two equations wrt to  $dT$  and  $dV$  respectively

$$U(T, V) = \frac{3}{2} nRT + f(V) \qquad U(T, V) = -\frac{n^2 a}{V} + g(T)$$

where  $f(V)$  and  $g(T)$  are constants of integration.

The only consistent solution is

$$U(T, V) = \frac{3}{2} nRT - \frac{n^2 a}{V} + \text{const} \qquad \text{QED}$$

B3) a)

i) Solution

The mean speed is given as

$$\bar{v} = \int_0^{\infty} v P(v) dv$$
$$\bar{v} = 4\pi \left[ \frac{m}{2\pi k_B T} \right]^{3/2} \int_0^{\infty} v^3 \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

We now use the standard integrals given to evaluate the integral which we recognize as  $I_3$  with

$\alpha = \frac{m}{2k_B T}$  and using  $I_3 = I_{2n+1}$  for  $n = 1$  and the standard integral

$$I_{2n+1} = \frac{n}{\alpha} I_{2n-1} = \frac{1}{\alpha} I_1 = \frac{1}{2\alpha^2} = \frac{1}{2} \left( \frac{2k_B T}{m} \right)^2$$

giving

$$\bar{v} = 4\pi \left[ \frac{m}{2\pi k_B T} \right]^{3/2} \frac{1}{2} \left[ \frac{2k_B T}{m} \right]^2 = 2 \left[ \frac{2k_B T}{\pi m} \right]^{1/2} = \left[ \frac{8k_B T}{\pi m} \right]^{1/2} \qquad \text{QED}$$

## Thermal & Kinetic Physics 2011

### Exam Solutions

#### ii) Solution

We are given the flux,  $\Phi = \frac{1}{4}n\bar{v}$  and number density  $n = \frac{N}{V}$

The equation of state for the ideal gas that we need is

$$PV = n_m RT = Nk_B T$$

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

$$n_{O_2} = 0.2 \times n = \frac{P}{5k_B T}$$

$$n_{O_2} = \frac{1.01 \times 10^5 \text{ Pa}}{5 \times 1.4 \times 10^{-23} \times 300 \text{ K}} = 4.8 \times 10^{24}$$

#### iii) Solution

First we need to find  $\bar{v}$  as follows

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}} = \sqrt{\frac{8 \times 1.4 \times 10^{-23} \text{ JK}^{-1} \times 300 \text{ K}}{3.14 \times 32 \times 1.66 \times 10^{-27} \text{ kg}}} = 2.0 \times 10^5 \text{ ms}^{-1}$$

Now we use  $\Phi = \frac{1}{4}n\bar{v} = \frac{1}{4} \times 4.8 \times 10^{24} \times 2 \times 10^5 \text{ m}^{-2} \text{ s}^{-1}$  is the number of oxygen molecules hitting every  $\text{m}^2$  of your lung per second and

$$\Phi = 2.4 \times 10^{29} \text{ m}^{-2} \text{ s}^{-1} = 2.4 \times 10^{25} \text{ cm}^2 \text{ s}^{-1}$$

b)

#### i) Solution

The ratio of fluxes incident upon the aperture of the pipe is

$$\frac{\Phi_1}{\Phi_2} = \frac{N_1^e}{N_2^e} = \frac{n_1 v_1}{n_2 v_2} = \frac{n_1}{n_2} \sqrt{\frac{m_2}{m_1}}$$

**QED**

## Thermal & Kinetic Physics 2011

### Exam Solutions

ii) Solution

$$\frac{N_{3\text{He}}^e}{N_{4\text{He}}^e} = \frac{n_{3\text{He}}}{n_{4\text{He}}} \sqrt{\frac{m_{4\text{He}}}{m_{3\text{He}}}} = \frac{10}{90} \sqrt{\frac{4}{3}} = 1.28 \times 10^{-1}$$

iii) Solution

*If this process is repeated through  $j$  cycles*

$$\frac{N_{3\text{He}}^e(j)}{N_{4\text{He}}^e} = \frac{10}{90} \left( \sqrt{\frac{4}{3}} \right)^j$$

*We require to find value for  $j$  that gives 30%  $^3\text{He}$*

$$\frac{N_{3\text{He}}^e(j)}{N_{4\text{He}}^e} = \frac{10}{90} \left( \sqrt{\frac{4}{3}} \right)^j = \frac{30}{70}$$

$$\left( \sqrt{\frac{4}{3}} \right)^j = \frac{30 \times 90}{70 \times 10}$$

$$j \ln \left( \sqrt{\frac{4}{3}} \right) = \ln \left( \frac{27}{7} \right)$$

$$j = \frac{\ln 3.86}{\ln 1.155} = \frac{1.351}{0.1437} = 9.4 \approx 10 \text{ cycles}$$

B4) a)

i) Solution

*The work is given by  $W = - \int_{V_i}^{2V_i} P dV = -P_i \int_{V_i}^{2V_i} dV = -P_i V_i$  and this is negative and therefore represents work done by the gas.*

ii) Solution

*The work is given by  $W = - \int_{V_i}^{V_i/2} P dV$*

*As it is adiabatic we have  $PV^\gamma = \text{const}$*

## Thermal & Kinetic Physics 2011

### Exam Solutions

$$W = -P_i V_i^\gamma \int_{V_i}^{V_i/2} V^{-\gamma} dV = \frac{-P_i V_i^\gamma}{1-\gamma} \left[ \left( \frac{V_i}{2} \right)^{1-\gamma} - V_i^{1-\gamma} \right]$$

$$W = \frac{-P_i V_i^\gamma}{1-\gamma} V_i^{1-\gamma} [2^{\gamma-1} - 1]$$

$$W = \frac{P_i V_i}{1-\gamma} (1 - 2^{\gamma-1})$$

$\gamma = \frac{C_P}{C_V}$  is always greater than 1 and therefore the denominator is negative as is the term in the brackets making  $W$  positive ie. work is done on the gas.

#### iii) Solution

The work is given by  $W = - \int_{V_i}^{V_i/2} P dV$

$$P = \frac{RT}{V}$$

$$W = -RT \int_{V_i}^{V_i/2} \frac{dV}{V} = -RT \ln \left( \frac{V_i/2}{V_i} \right)$$

$$W = -RT \ln \left( \frac{1}{2} \right) = RT \ln 2$$

$W$  is positive and therefore represents work done on the gas.

b)

#### i) Solution

In an isobaric process we use the heat capacity at constant pressure to find  $Q_1$

$$Q_1 = Q_{b \rightarrow c} = C_P (T_c - T_b)$$

#### ii) Solution

In an isochoric process we use the heat capacity at constant volume to find  $Q_2$

$$Q_2 = -Q_{d \rightarrow a} = C_V (T_d - T_a)$$

## Thermal & Kinetic Physics 2011

### Exam Solutions

#### iii) Solution

The engine efficiency is given by  $\eta_E = 1 - \frac{Q_2}{Q_1}$  and using the results from i) and ii)

$$\eta_E = 1 - \frac{C_V(T_d - T_a)}{C_P(T_c - T_b)} = 1 - \frac{1}{\gamma} \frac{(T_d - T_a)}{(T_c - T_b)} \quad \underline{QED}$$

c)

#### i) Solution

Considering the two adiabatic processes  $c \rightarrow d$  and  $a \rightarrow b$  and using the adiabatic rule  $c \rightarrow d$  is an adiabatic process thus

$$T_c V_c^{\gamma-1} = T_d V_d^{\gamma-1} \quad \Rightarrow \quad T_d = T_c \left( \frac{V_c}{V_d} \right)^{\gamma-1}$$

$a \rightarrow b$  is also an adiabatic process and similarly

$$T_a V_a^{\gamma-1} = T_b V_b^{\gamma-1} \quad \Rightarrow \quad T_a = T_b \left( \frac{V_b}{V_a} \right)^{\gamma-1}$$

We also know that

$$P_b V_b = nRT_b \quad P_c V_c = nRT_c \quad \text{and} \quad P_b = P_c$$

$$\frac{V_b}{V_c} = \frac{T_b}{T_c} \quad \Rightarrow \quad T_c = T_b \frac{V_c}{V_b}$$

$$T_d - T_a = T_c \left( \frac{V_c}{V_a} \right)^{\gamma-1} - T_b \left( \frac{V_b}{V_a} \right)^{\gamma-1}$$

$$T_d - T_a = T_b \left( \frac{V_c}{V_b} \right) \left( \frac{V_c}{V_a} \right)^{\gamma-1} - T_b \left( \frac{V_b}{V_a} \right)^{\gamma-1} = \frac{T_b V_a}{V_b} \left[ \left( \frac{V_c}{V_a} \right)^{\gamma} - \left( \frac{V_b}{V_a} \right)^{\gamma} \right]$$

And similarly

$$T_c - T_b = T_b \left( \frac{V_c}{V_b} \right) - T_b = \frac{T_b V_a}{V_b} \left[ \left( \frac{V_c}{V_a} \right) - \left( \frac{V_b}{V_a} \right) \right]$$

## Thermal & Kinetic Physics 2011

### Exam Solutions

Including these in the earlier expression for  $\eta_E$

$$\eta_E = 1 - \frac{1}{\gamma} \frac{T_d - T_a}{T_c - T_b} = 1 - \frac{1}{\gamma} \frac{\left[ \left( \frac{V_a}{V_c} \right)^{-\gamma} - \left( \frac{V_a}{V_b} \right)^{-\gamma} \right]}{\left( \frac{V_a}{V_c} \right)^{-1} - \left( \frac{V_a}{V_b} \right)^{-1}}$$

We have  $\gamma_e = \frac{V_a}{V_c}$  is the expansion ratio and  $\gamma_c = \frac{V_a}{V_b}$  is the compression ratio and when these are used

in the above

$$\eta_E = 1 - \frac{1}{\gamma} \left[ \frac{\gamma_e^{-\gamma} - \gamma_c^{-\gamma}}{\gamma_e^{-1} - \gamma_c^{-1}} \right] \quad \underline{\underline{QED}}$$

ii) Solution

For a rigid diatomic gas  $\gamma = \frac{C_P}{C_V} = \frac{7}{5}$

Also we are given  $\gamma_e = \frac{V_a}{V_c} = \frac{5000}{3000} = 1.666$  and  $\gamma_c = \frac{V_a}{V_b} = \frac{5000}{500} = 10$

$$\eta_E = 1 - \frac{1}{\gamma} \left[ \frac{\gamma_e^{-\gamma} - \gamma_c^{-\gamma}}{\gamma_e^{-1} - \gamma_c^{-1}} \right] = 1 - \frac{5}{7} \left[ \frac{1.666^{-1.4} - 10^{-1.4}}{1.666^{-1} - 10^{-1}} \right]$$

$$\eta_E = 1 - 0.71 \times \left[ \frac{0.49 - 3.98 \times 10^{-2}}{6 \times 10^{-1} - 1 \times 10^{-1}} \right] = 1 - 0.71 \times \frac{0.45}{0.5} = 1 - 0.639 = 0.361$$

iii) Solution

The efficiency of a heat pump is given by

$$\eta_{HP} = \frac{Q_1}{W} = \frac{1}{\eta_E} = \frac{1}{0.361} = 2.77$$

$$\dot{Q}_1 = 2.77 \times W = 2.77 \times 3 \times 10^3 \text{ Js}^{-1} = 8.31 \times 10^3 \text{ Js}^{-1}$$

and

## Thermal & Kinetic Physics 2011

### Exam Solutions

$$\eta_{HP} = \frac{Q_1}{Q_1 - Q_2} = \frac{\dot{Q}_1}{\dot{Q}_1 - \dot{Q}_2} = 2.77$$

$$2.77 \dot{Q}_2 = (2.77 - 1) \dot{Q}_1$$

$$\dot{Q}_2 = \frac{1.77}{2.77} \dot{Q}_1 = 5.31 \times 10^3 \text{ Js}^{-1}$$