

**Thermal & Kinetic Physics 2010
Solutions**

**Thermal & Kinetic Physics (PHY-214) Exam 2010
Solutions.**

SECTION A

A1. $dU = TdS - PdV$

dU is the incremental change in internal energy.

dS is the incremental change in entropy.

$-PdV$ is the incremental work done on the system where P is the pressure and dV the incremental change in volume. [5]

A2. $S = k_B \ln \Omega$ [5]

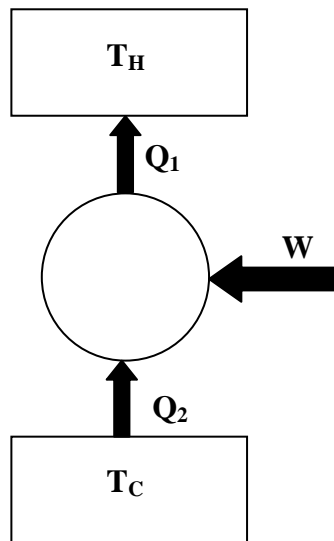
A3. $\eta = 1 - \frac{T_2}{T_1}$ [5]

A4. $\Delta S \geq 0$
where $\Delta S = 0$ for a reversible process. [5]

A5.

$\eta_R = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$

or $\frac{1}{\frac{Q_1}{Q_2} - 1}$



A6. $P_i V_i^\gamma = P_f V_f^\gamma$ where $\gamma = \frac{c_P}{c_V}$ is the ratio of the specific heat capacity at constant pressure, c_P to that at constant volume, c_V . [5]

A7. $C_P = \left(\frac{\partial H}{\partial T} \right)_P$

H is the enthalpy, T the temperature and P the pressure of the system. [5]

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A8.

$$U = \frac{5}{2} Nk_B T \quad [3]$$

$$\left\langle \frac{1}{2} mv^2 \right\rangle = \frac{3}{2} k_B T \quad [2]$$

A9. For an ideal gas $PV = RT$ and therefore we can write

$$P = \frac{RT}{V} \quad [1]$$

$$\left(\frac{\partial P}{\partial V} \right)_T = \frac{-RT}{V^2} \quad [1]$$

$$\kappa = -V \left(\frac{\partial P}{\partial V} \right)_V = \frac{RT}{V} = P \quad [3]$$

A10.

$$dW = -PdV$$

$$\Delta W = - \int_i^f PdV = -P \int_{V_i}^{V_f} dV = P(V_i - V_f) = +P \frac{V}{2} \quad [4]$$

The sign is positive and therefore represents work done ON the gas. [1]

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SECTION B

B1

a)

i)
$$\Delta W = -\int_i^f PdV$$

$$PV = RT \qquad P = \frac{RT}{V}$$

$$\Delta W = -\int_i^f PdV = -RT \int_{V_i}^{V_f} \frac{dV}{V} = -RT(\ln V_f - \ln V_i) = -RT \ln 2$$

As $\ln 2$ is greater than 0 it follows that $\Delta W < 0$ and this represents work done BY the gas. [2]

ii) In an isochoric process there is no work done and therefore by the first law of thermodynamics we have

$$\Delta U = \Delta Q = \frac{3}{2}V_1\left(\frac{P_1}{2} - P_1\right) = -\frac{3}{4}P_1V_1$$

This is a negative quantity indicating that heat has been released by the gas. [1]

iii)
$$\Delta U = \frac{5}{2}P_1\left(\frac{V_1}{2} - V_1\right) = -\frac{5}{4}P_1V_1$$
 [1]

b)

i) From the adiabatic rule $P_iV_i^\gamma = P_fV_f^\gamma$ by using $PV = RT$ and $P = \frac{RT}{V}$ we obtain the rule in terms of V and T

$$T_iV_i^{\gamma-1} = T_fV_f^{\gamma-1}$$

$$T_f = T_i\left(\frac{V_i}{V_f}\right)^{\gamma-1} = 298.15 \times 2^{0.4} = 393.41K = 144.02 \text{ } ^\circ C \text{ } _$$
 [2]

ii)
$$U = \frac{5}{2}RT$$

$$\Delta U = \frac{5}{2}R\Delta T$$

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$$\Delta U = \frac{5}{2} 8.31 J \text{ mol}^{-1} K^{-1} \times 62.27 K = +1293.7 J \quad [2]$$

iii) There is no heat transfer during an adiabatic process and the first law of thermodynamics then gives

$$\Delta W = \Delta U = +1293.7 J$$

Being positive the work is done on the gas during this process. [2]

c)

i) $\eta_E = 1 - \frac{Q_2}{Q_1}$ [1]

ii) To find the engine efficiency we need to begin by finding Q_1 and Q_2 .

To find Q_1 begin with the fact that **2** \rightarrow **3** is an isochore and no work is done as $dV = 0$. The heat can be calculated from the first law

There are two ways to proceed either is acceptable

$$Q_1 = \Delta Q_{32} = \Delta U_{32} = U_3 - U_2 = \frac{5}{2} nR(T_3 - T_2)$$

or alternatively

$$Q_1 = \Delta Q_{32} = \Delta U_{32} = \frac{5}{2} V_2 (P_3 - P_2)$$

We can see that Q_1 is positive as $T_3 > T_2$ ($P_3 > P_2$) and it is therefore a flow of heat into the gas.

To find Q_2 we may use the fact that **4** \rightarrow **1** is also an isochore and again no work is done. The heat can be calculated from the first law as before

$$\Delta Q_{14} = \Delta U_{14} = U_1 - U_4 = \frac{5}{2} nR(T_1 - T_4)$$

Or alternatively

$$\Delta Q_{14} = \Delta U_{14} = \frac{5}{2} V_1 (P_1 - P_4)$$

ΔQ_{14} is negative as $T_1 < T_4$ ($P_1 < P_4$) and it is therefore a flow of heat from the gas as shown. However we need the modulus of ΔQ_{14} to represent Q_2 in the expression for

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engine efficiency as this expression requires a positive value (the direction of heat flow is implicitly accounted for in the expression for efficiency and thus

$$Q_2 = -\Delta Q_{14} = \frac{5}{2}nR(T_4 - T_1)$$

Or

$$Q_2 = -\Delta Q_{14} = \frac{5}{2}V_1(P_4 - P_1)$$

Using our earlier expression for the efficiency

$$\eta_E = 1 - \frac{Q_2}{Q_1} = 1 - \frac{(T_4 - T_1)}{(T_3 - T_2)}$$

Or

$$\eta_E = 1 - \frac{Q_2}{Q_1} = 1 - \frac{V_1(P_4 - P_1)}{V_2(P_3 - P_2)}$$

We need this in terms of the two volumes V_1 and V_2 and to achieve this use the fact that **1** \rightarrow **2** is an adiabatic compression where work is done on the gas and the equation from **b i)** holds;

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$$

Or

$$P_1V_1^\gamma = P_2V_2^\gamma$$

Also **3** \rightarrow **4** is an adiabatic expansion where work is done by the gas

$$T_4V_4^{\gamma-1} = T_3V_3^{\gamma-1}$$

Or

$$P_4V_4^\gamma = P_3V_3^\gamma$$

Subtracting the first of these adiabatic expressions from the second and noting that $V_1 = V_4$ and that $V_2 = V_3$

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$$(T_4 - T_1)V_1^{\gamma-1} = (T_3 - T_2)V_2^{\gamma-1}$$

Or

$$(P_4 - P_1)V_1^\gamma = (P_3 - P_2)V_2^\gamma$$

therefore

$$\frac{(T_4 - T_1)}{(T_3 - T_2)} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

Or

$$\frac{(P_4 - P_1)}{(P_3 - P_2)} = \left(\frac{V_2}{V_1}\right)^\gamma$$

and

$$\eta_E = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

$$\eta_E = 1 - \frac{1}{r^{\gamma-1}}$$

QED [6]

iii) The compression ratio is

$$r = \frac{V_1}{V_2} = \frac{300\text{cm}^3}{75\text{cm}^3} = 4 \quad [2]$$

And therefore

$$1. \quad \eta_E = 1 - \frac{1}{r^{\gamma-1}} = 1 - \frac{1}{4^{0.4}} = 0.425 \quad [2]$$

$$2. \quad P_2 = P_1 \left(\frac{V_1}{V_2}\right)^\gamma = P_1 r^{1.4} = 1 \text{ atm} \times 4^{1.4} = 6.96 \text{ atm} \quad [2]$$

$$3. \quad Q_1 = \frac{5}{2} nR(T_3 - T_2) = \frac{5}{2} (P_3 V_3 - P_2 V_2) = \frac{5}{2} V_2 (P_3 - P_2) \quad [2]$$

$$Q_1 = \frac{5}{2} V_2 (P_3 - P_2) = \frac{5}{2} \times 75 \times 10^{-6} \text{ m}^3 (25 - 6.96) \times 1.01 \times 10^5 \text{ Pa} = 341.62 \text{ J}$$

4. Finally if $\eta_E = 0.425$ and $Q_1 = 341.62 \text{ J}$ we have

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$$W = \eta_E Q_1 = 145 \text{ J per cycle}$$

Therefore with 50 cycles per second

$$\text{Power} = 50 \times 145 \text{ Watts} = 7.26 \text{ kW} \quad [2]$$

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B2.

a)

i)
$$\Delta S = S_f - S_i = \int_i^f \frac{\delta Q_R}{T}$$

The reversible heat transfer δQ_R is by convention positive when heat flows into the system whose entropy change is being considered. [2]

ii) The entropy change of the water as it cools is

$$\Delta S = \int_i^f \frac{\delta Q_R}{T} \text{ and } \delta Q_R = mC_P dT$$

Therefore

$$\Delta S = mC_P \int_{303.15}^{280.15} \frac{dT}{T} = 3\text{kg} \times 4.2 \times 10^3 \text{ Jkg}^{-1}\text{K}^{-1} \times \ln \frac{280.15}{303.15}$$

$$\Delta S = -994 \text{ JK}^{-1} \quad [2]$$

iii) This process occurs in three stages and the calculation must be split into three parts;

1) The ice warming from -10°C to 0°C

$$\Delta S_1 = \int_i^f \frac{\delta Q_R}{T} \text{ and } \delta Q_R = mC_P dT$$

$$\Delta S_1 = mC_P^{Ice} \int_{263.15}^{273.15} \frac{dT}{T} = 3\text{kg} \times 2.1 \times 10^3 \text{ Jkg}^{-1}\text{K}^{-1} \times \ln \frac{273.15}{263.15}$$

$$\Delta S_1 = 236.0 \text{ JK}^{-1}$$

2) The ice melting at constant temperature

$$\Delta S_2 = \frac{ml_{melt}}{T} = \frac{3\text{kg} \times 3.33 \times 10^5 \text{ Jkg}^{-1}}{273.15}$$

$$\Delta S_2 = 3657.33 \text{ JK}^{-1}$$

3) The melted ice water warming up to ambient temperature

$$\Delta S_3 = mC_P \int_{273.15}^{280.15} \frac{dT}{T} = 3\text{kg} \times 4.2 \times 10^3 \text{ Jkg}^{-1}\text{K}^{-1} \times \ln \frac{280.15}{273.15}$$

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$$\Delta S_3 = 319.0 \text{ JK}^{-1}$$

The net change in entropy is the sum of these three

$$\Delta S_{net} = 4212.33 \text{ JK}^{-1} \quad [5]$$

b)

i) We have the thermodynamic identity for a gas

$$dU = TdS - PdV$$

Also, for a monatomic ideal gas

$$U = \frac{3}{2} RT$$

Therefore

$$dU = \frac{3}{2} RdT$$

Re-arranging the expression for the thermodynamic identity

$$TdS = dU + PdV = PdV + \frac{3}{2} RdT$$

From the equation of state for an ideal gas

$$P = \frac{RT}{V}$$

$$TdS = RT \frac{dV}{V} + \frac{3}{2} RdT$$

Re-arranging

$$dS = R \frac{dV}{V} + \frac{3}{2} R \frac{dT}{T}$$

Integrating both sides at constant volume

$$\Delta S = R \frac{3}{2} \ln T + f(V)$$

Integrating both sides at constant temperature

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$$\Delta S = R \ln T + g(T)$$

For both to be generally true in passing from initial temperature and volume, T_i, V_i to final temperature and volume, T_f, V_f

$$\Delta S = \frac{3}{2} R \ln \left[\frac{T_f}{T_i} \right] + R \ln \left[\frac{V_f}{V_i} \right] \quad \text{QED [5]}$$

ii) In an isothermal process we have;

$$\Delta S = R \ln \frac{V_f}{V_i} = R \ln \frac{1}{3} = -8.31 \times 1.1 = -9.13 \text{ JK}^{-1} \quad [2]$$

iii) In an isochoric process we have

$$\Delta S = \frac{3}{2} R \ln \left[\frac{T_f}{T_i} \right] = 1.5 \times 8.31 \times \ln 2 = +8.63 \text{ JK}^{-1} \quad [2]$$

c)

i) The system will reach equilibrium by reaching the temperature of the reservoir and in this process

$$\Delta S_{\text{system}} = \int_{T_i}^{T_R} \frac{\delta Q}{T} = m C_P \int_{T_i}^{T_R} \frac{dT}{T} = m C_P \ln \left[\frac{T_R}{T_i} \right]$$

And the reservoir remains at a constant temperature giving up heat ΔQ to the system and therefore

$$\Delta S_{\text{reservoir}} = \int_i^f \frac{dQ_R}{T} = \frac{1}{T_R} \int_i^f dQ_R = \frac{\Delta Q_R}{T_R} = -\frac{m C_P}{T_R} (T_R - T_i)$$

$$\Delta S_{\text{Universe}} = \Delta S_{\text{System}} + \Delta S_{\text{reservoir}}$$

$$\Delta S_{\text{Universe}} = m C_P \ln \left(\frac{T_R}{T_i} \right) - m C_P \left(1 - \frac{T_i}{T_R} \right) = -m C_P \ln \left(\frac{T_i}{T_R} \right) - m C_P \left(1 - \frac{T_i}{T_R} \right)$$

which can be re-written as

$$\Delta S_{\text{Universe}} = -m C_P \ln \left[1 - \left(1 - \frac{T_i}{T_R} \right) \right] - m C_P \left(1 - \frac{T_i}{T_R} \right)$$

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This is in the form

$$\Delta S_{\text{Universe}} = -mC_P \ln(1-X) - mC_P X$$

where

$$X = \left(1 - \frac{T_i}{T_R}\right) \quad \text{QED [5]}$$

- ii) For the system warming, $\frac{T_i}{T_R} < 1$ and X is positive.

Using McLauriens series expansion, $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$$\Delta S_{\text{Universe}} = mC_P \left[X + \frac{X^2}{2} + \frac{X^3}{3} + \dots - X \right] = mC_P \left[\frac{X^2}{2} + \frac{X^3}{3} + \dots \right] > 0$$

$\Delta S_{\text{Universe}}$ has increased.

QED [2]

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B3.

a)

i) Starting with the ideal gas equation of state

$$PV = RT$$

The three variables are P, V, T and we form the cyclic relation

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1$$

$$P = \frac{RT}{V}$$

$$V = \frac{RT}{P}$$

$$T = \frac{PV}{R}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{-RT}{V^2}$$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{R}{P}$$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{V}{R}$$

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = \frac{-RT}{V^2} \frac{R}{P} \frac{V}{R} = \frac{-PV}{V^2} \frac{RV}{PR} = -1 \quad \mathbf{QED} \quad [2]$$

ii) The product $\kappa\beta$ may be written using the given definitions of the two quantities as

$$\kappa\beta = -\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P$$

We have already from part i) using the cyclical relation

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1$$

Therefore using the reciprocal relation

$$\kappa\beta = -\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{\left(\frac{\partial T}{\partial P}\right)_V} = \left(\frac{\partial P}{\partial T}\right)_V$$

The required differential relationship.

[2]

iii) The pressure is changed isothermally and therefore use $\kappa = -V\left(\frac{\partial P}{\partial V}\right)_T$ to find the

change in temperature by considering small changes

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$$\kappa = -V \left(\frac{\partial P}{\partial V} \right)_T = -\frac{V}{\Delta V} \Delta P$$

We also have from part i) $\left(\frac{\partial P}{\partial V} \right)_T = \frac{-RT}{V^2} \Rightarrow \left(\frac{\partial P}{\partial V} \right)_T = \frac{-PV}{V^2}$

Therefore

$$\kappa = -V \left(\frac{\partial P}{\partial V} \right)_T = P = -\frac{V}{\Delta V} \Delta P$$

And

$$\frac{V}{\Delta V} = -\frac{P}{\Delta P} = -0.02$$

The volume also changes decreasing by -2%. [2]

b)

i) Given the definition of the Gibbs free energy, $G = U - TS + PV$

To find the natural variables we write the infinitesimal of G

$$dG = dU - TdS - SdT + PdV + VdP$$

Now use the thermodynamic identity, $dU - TdS + PdV = 0$ to tidy this up

$$dG = -SdT + VdP$$

The natural variables of G are then T and P

$$G = G(T, P) \tag{4}$$

ii) Using the natural variables we may write the infinitesimal of G as

$$dG = \left(\frac{\partial G}{\partial T} \right)_P dT + \left(\frac{\partial G}{\partial P} \right)_T dP$$

And compare this with the expression of dG from i)

$$\left(\frac{\partial G}{\partial T} \right)_P = -S \quad \text{and} \quad \left(\frac{\partial G}{\partial P} \right)_T = V \tag{3}$$

iii) From i) and ii) and using the fact that dG is a perfect differential and that T and P are the natural variables of G gives

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$$\frac{\partial}{\partial P} \left(\frac{\partial G}{\partial T} \right)_P = \frac{\partial}{\partial T} \left(\frac{\partial G}{\partial P} \right)_T$$

And therefore

$$-\frac{\partial S}{\partial P} = \frac{\partial V}{\partial T}$$

Which is the required Maxwell relation.

[3]

c)

i) In part a the thermal expansion coefficient was defined as

$$\beta = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P$$

We are given the equation of state for one mole of a van der Waals gas,

$$\left(P + \frac{a}{V^2} \right) (V - b) = RT$$

We need to find $\left(\frac{\partial V}{\partial T} \right)_P$

Begin by finding $\left(\frac{\partial T}{\partial V} \right)_P$

$$\left(\frac{\partial T}{\partial V} \right)_P = \frac{1}{R} \left(\frac{-2a}{V^3} (V - b) + P + \frac{a}{V^2} \right)$$

$$\left(\frac{\partial T}{\partial V} \right)_P = \frac{1}{R} \left(\frac{-2a}{V^2} + \frac{2ab}{V^3} + P + \frac{a}{V^2} \right)$$

We need to use the reciprocal relation and therefore to invert this equation and start by getting the RHS over a common denominator

$$\left(\frac{\partial T}{\partial V} \right)_P = \frac{1}{R} \left(\frac{-2aV}{V^3} + \frac{2ab}{V^3} + \frac{PV^3}{V^3} + \frac{aV}{V^3} \right) = \frac{PV^3 - aV + 2ab}{RV^3}$$

$$\left(\frac{\partial V}{\partial T} \right)_P = \frac{RV^3}{PV^3 - aV + 2ab}$$

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And therefore following the definition given we have

$$\beta = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P = \frac{RV^2}{PV^3 - aV + 2ab} \quad [4]$$

ii) Given that the internal energy of a van der Waals gas is;

$$U_{vdW} = \frac{3}{2}RT - \frac{a}{V}$$

We can obtain dU and use this in the first law

$$dU = \frac{3}{2}RdT + \frac{a}{V^2}dV$$

The first law is

$$dU = \delta Q - PdV$$

And for an adiabatic process

$$dU = -PdV$$

therefore

$$\frac{3}{2}RdT + \frac{a}{V^2}dV = -PdV$$

From the van der Waals equation of state, $\left(P + \frac{a}{V^2} \right)(V - b) = RT$

$$P = \frac{RT}{V - b} - \frac{a}{V^2}$$

Using this expression for P

$$\frac{3}{2}RdT + \frac{a}{V^2}dV = \frac{-RT}{V - b}dV + \frac{a}{V^2}dV$$

$$\frac{3}{2}RdT = \frac{-RT}{V - b}dV$$

$$\frac{3}{2} \frac{dT}{T} = \frac{-dV}{V - b}$$

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Integrating both sides

$$\frac{3}{2} \ln T = \ln T^{3/2} = -\ln(V - b) + \text{const}$$

$$\ln T^{3/2} + \ln(V - b) = \text{const}$$

$$T^{3/2}(V - b) = \text{const}$$

QED [5]

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B4

a)

i)
$$v_{rms} = \sqrt{\frac{3k_B T}{m}} \quad [2]$$

ii) Using the equation from part i) we find the room mean square velocity of N₂

$$v_{rms} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \text{ JK}^{-1} \times 300 \text{ K}}{28 \times 1.66 \times 10^{-27}}}$$
$$v_{rms}(N_2) = \sqrt{2.67 \times 10^5} = 517 \text{ ms}^{-1} \quad [2]$$

Noting that the velocity depends on $\sqrt{m^{-1}}$ and that the mass of O₂ is a factor of $\frac{32}{28} = 1.1428$ larger, the velocity will be a factor 1.069 smaller

$$v_{rms}(O_2) = \frac{517 \text{ ms}^{-1}}{1.069} = 483.6 \text{ ms}^{-1} \quad [2]$$

b)

i) The Maxwell speed distribution is $P(v)dv = 4\pi v^2 f(v)dv$

$$P(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 \exp\left[-\frac{mv^2}{2k_B T}\right] \quad [2]$$

ii) To find the mean of a quantity requires integration of the product of that quantity with the probability distribution over all possible values;

$$\bar{v} = \int_0^{\infty} v P(v) dv \quad [2]$$

iii) Making the suggested substitution

$$v^2 \rightarrow x, \quad dv = \frac{dx}{2v}$$

and writing $a = \frac{m}{2k_B T}$ the integral of part ii) becomes

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$$\bar{v} = 4\pi \left[\frac{a}{\pi} \right]^{3/2} \frac{1}{2} \int_0^{\infty} x \exp(-ax) dx = 2\pi \left[\frac{a}{\pi} \right]^{3/2} \int_0^{\infty} x \exp(-ax) dx$$

We can find the integral, $\int_0^{\infty} x \exp(-ax) dx$, by integrating by parts; $\int u dv = uv - \int v du$

as follows;

$$u = x \qquad dv = e^{-ax}$$

$$du = dx \qquad v = -\frac{1}{a} e^{-ax}$$

$$\int u dv = \int x e^{-ax} = -\frac{x}{a} e^{-ax} - \int -\frac{1}{a} e^{-ax} dx$$

$$\int_0^{\infty} x e^{-ax} dx = \left[-\frac{x}{a} e^{-ax} - \frac{1}{a^2} e^{-ax} \right]_0^{\infty} = \frac{1}{a^2}$$

$$\bar{v} = 2\pi \left[\frac{a}{\pi} \right]^{3/2} \frac{1}{a^2} = \sqrt{\frac{4}{\pi a}} = \sqrt{\frac{8k_B T}{\pi m}} \quad [6]$$

c)

i) We are given the expression for the flux, $\Phi = \frac{1}{4} n \bar{v}$ where $n = \frac{N}{V}$.

We may get the flux in terms of temperature and pressure by using the ideal gas equation of state,

$$PV = Nk_B T$$

Therefore

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

And the temperature will enter also through the mean velocity as found in part i),

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}}$$

Therefore the flux is

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$$\Phi(P, T) = \frac{1}{4} \frac{P}{k_B T} \sqrt{\frac{8k_B T}{\pi m}} = \frac{P}{\sqrt{2\pi m k_B T}} \quad [3]$$

ii) In the effusion process described every gas molecule incident upon the hole will escape. Thus the number that escape per unit time of molecular species i is given by,

$$N_i^e = \Phi_i A = \frac{1}{4} n_i \sqrt{\frac{8k_B T}{\pi m_i}}$$

Therefore

$$\frac{N_1^e}{N_2^e} = \frac{n_1}{n_2} \sqrt{\frac{m_2}{m_1}} \quad \text{QED} \quad [3]$$

iii) From part ii) we find the ratio of Helium isotopes as

$$\frac{N_{3He}^e}{N_{4He}^e} = \frac{n_{3He}}{n_{4He}} \sqrt{\frac{m_{4He}}{m_{3He}}} = \frac{10}{90} \sqrt{\frac{4}{3}} = 0.128$$

This tells us that $N_{3He}^e = 0.128 N_{4He}^e$ and therefore that

$N_{3He}^e + N_{4He}^e = 1.128 N_{4He}^e$ so the percentage of ${}^3\text{He}$ in the escaped mixture is

$$\frac{N_{3He}^e}{N_{3He}^e + N_{4He}^e} \times 100 = \frac{0.128}{1.128} \times 100 = 11.4\% \quad [3]$$