Queen Mary
University of London

M. Sc. Examination by course unit 2009

ASTM001 Solar System

Duration: 3 hours

Date and time: 28 May 2009, 1815h–2115h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): J.R. Donnison, D.H. Burgess

Useful relations:

- Gradient in polar coordinates $\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}$
- Velocity in polar coordinates $\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\theta}$

Section A: Each question carries 5 marks . You should attempt ALL five questions and give definitions where appropriate.

Question 1 Describe briefly (a few sentences) what is meant by

(a) Kepler's laws of Planetary Motion

Question 2 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) The restricted three-body problem
(b) Lagrangian points

Question 3 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Tisserand's Constant
(b) Hill Sphere

Question 4 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Zero velocity surfaces
(b) Tidal dissipation factor Q

Question 5 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Minimum Mass Nebula
(b) Planetary differentiation

Section B: Each question carries 25 marks. There are 4 questions. You may attempt all questions, but only marks for the best 3 questions will be counted.

Question 6 Two bodies with masses m_1 and m_2 move under their mutual gravitational attraction. The equation of motion defining the variation of the position vector $\mathbf{r}$ of the mass m_2 with respect to the mass m_1 is

$$\ddot{\mathbf{r}} + \mathcal{G}(m_1 + m_2)\frac{\mathbf{r}}{r^3} = 0\,,$$

where $\mathcal{G}$ is the universal gravitational constant.

(a) Taking the vector product of $\mathbf{r}$ with the above equation and using the standard result, $\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\theta}$, for motion in a polar coordinate system, show that $r^2\dot{\theta} = h$, where h is a constant. [4]

(b) By taking the scalar product of the same equation of motion with the velocity vector, $\dot{\mathbf{r}}$, and integrating, show that

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C$$

where $v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, and C is a constant of the motion. [5]

(c) In a polar coordinate system, the acceleration vector is given by

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\right]\hat{\theta}\,.$$

Use the fact that the value of h defined in part (a) is a constant to show that this equation of motion can be written as the scalar equation

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}\,,$$

where $\mu = \mathcal{G}(m_1 + m_2)$. Use the substitution, $u = 1/r$, to derive expressions for $\dot{r}$ and $\ddot{r}$ in terms of u, θ and h and hence show that the equation of motion can be written as

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\,.$$

[8]

(d) Given that m_2 is a comet and m_1 is the Sun, so that $m_1 \ll m_2$ write down the general solution to this differential equation. Given also that the comet is initially projected with a velocity V from infinity towards the Sun on a hyperbolic orbit, show that if it passes the Sun at its closest approach with velocity U when θ is zero, then the path of the comet is given by

$$\left(\frac{4\mu U^2}{(U^2 - V^2)^2}\right)\frac{1}{r} = 1 + \left(\frac{U^2 + V^2}{U^2 - V^2}\right)\cos\theta\,.$$

[8]

Question 7 In the planar, circular restricted three-body problem the equations of motion of the massless test particle in the frame rotating with unit angular velocity are given by

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial U}{\partial y}, \end{aligned}$$

where the test particle has rectangular coordinates (x, y), in a frame where the x-axis is directed along the line joining the two masses, and

$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

with

$$\mu_1 = m_1/\left(m_1 + m_2\right), \mu_2 = m_2/\left(m_1 + m_2\right), \mu_1 + \mu_2 = 1,$$

and it is assumed that $m_2 < m_1$. The distances of the particle to the masses m_1 and m_2 are given respectively by

$$r_1 = \sqrt{\left(x + \mu_2\right)^2 + y^2} \text{ and } r_2 = \sqrt{\left(x - \mu_1\right)^2 + y^2}.$$

The unit of distance is taken to be separation of the two masses.

(a) Show that

$$\mu_1 r_1^2 + \mu_2 r_2^2 = x^2 + y^2 + \mu_1 \mu_2,$$

hence rewrite U in terms of r_1 and r_2 only. [6]

(b) State the conditions for determining the equilibrium points and hence deduce that there are equilibrium points where $r_1 = r_2 = 1$. Find the x and y positions of the equilibrium points and sketch their position. [9]

(c) An equilibrium point is located close to m_2 but outside it beyond the line joining the two masses such that $r_1 - r_2 = 1$. Using the new expression for U, show that this equilibrium point r_2 must satisfy the equation.

$$\frac{\mu_2}{\mu_1} = \frac{3r_2^3\left(1 + r_2 + r_2^2/3\right)}{\left(1 + r_2\right)^2\left(1 - r_2^3\right)}$$

[6]

(d) Given that $m_1 \gg m_2$ and $r_2 \ll 1$, find the approximate distance of the equilibrium point.What does this distance represent? [4]

Question 8 The tidal potential per unit mass experienced by a satellite of mass m and semi-major axis a due to the tidal bulge it raises on a homogeneous planet of radius R and mass M is

$$V = -k_2 G \frac{m}{a^6} R^5 P_2(\cos\theta)\,,$$

where k_2 is the Love number of the planet, G is the universal gravitational constant, θ is the lag angle, and $P_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ is the Legendre polynomial of degree 2.

(a) Show that the tangential component of the force, that is in the θ direction, due to this potential is

$$F = \frac{3}{2} G \frac{m^2}{a^7} R^5 k_2 \sin 2\theta\,.$$

Hence find the resulting torque on the satellite. [5]

(b) By considering the sum of the rotational energy of the planet and the orbital energy of the satellite-planet system, show that the rate of change of energy is given by

$$\dot{E} = I\Omega\dot{\Omega} + \frac{1}{2}\left(\frac{Mm}{M+m}\right) n^2 a\dot{a},$$

where I is the moment of inertia of the planet, Ω is the rotational frequency of the planet, and n is the mean motion of the satellite. [5]

(c) Use the conservation of the total angular momentum of the system and the result from (b) to show that

$$\dot{E} = -\frac{1}{2}\left(\frac{Mm}{M+m}\right) an\dot{a}(\Omega - n)\,.$$

[7]

(d) Given that $\dot{E} = \Gamma(\Omega - n) < 0$, use the results from (a), (b), and (c) to show that if $Q = 1/\sin 2\theta$ is the tidal dissipation function of the planet and $M \gg m$, then if m starts from a distance a_0 it will impact M after a time

$$t_{impact} = \frac{2}{39}\left(\frac{MR^{13}}{G}\right)^{\frac{1}{2}} \frac{Q}{mR^5}\left(\left(\frac{a_0}{R}\right)^{\frac{13}{2}} - 1\right).$$

[8]

Question 9 (a) Show that for a sufficiently massive body of mass M and radius R, where gravitational focussing is important that the mass accretion rate is

$$\frac{dM}{dt} = \rho\pi R^2 \left(1 + \left(\frac{v_{esc}}{v_r}\right)^2\right) v_r,$$

where ρ is the mass density of the accreted material and v_r is the relative velocity with which the accreting material approaches M. [10]

(b) Consider the two cases

(i) $v_{esc} \gg v_r$,
(ii) $v_{esc} \ll v_r$,

and discuss them in terms of the growth of M. [8]

(c) Given that the resulting planetary body has mass M and radius R and is spherical with uniform density, show that the resulting gravitational binding energy of the body is given by

$$E = -\frac{3}{5}\frac{GM^2}{R}.$$

[7]

End of Paper