Solutions HW5

1) $\psi \to \psi' = P\psi = \gamma^0 \psi \qquad \gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$

a) $$U(\vec{p}, s) = \begin{pmatrix} \phi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{(E+m)} \phi_s \end{pmatrix} \sqrt{E+m} \qquad p^\mu = (E, \vec{p})$$

$$P\,U(\vec{p}, s) = \gamma^0 U(\vec{p}, s) = \sqrt{E+m} \begin{pmatrix} \phi_s \\ -\frac{\vec{\sigma}\cdot\vec{p}}{(E+m)} \phi_s \end{pmatrix} = U(-\vec{p}, s)$$
$$\equiv U((E, -\vec{p}), s)$$

b) $$V(\vec{p}, s) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{(E+m)} \chi_s \\ \chi_s \end{pmatrix}$$

$$PV = \gamma^0 V(p, s) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi_s \\ -\chi_s \end{pmatrix} =$$

$$= -\sqrt{E+m} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi_s \\ \chi_s \end{pmatrix} = -V(-\vec{p}, s)$$

c) $$C = i\gamma^2\gamma^0 = i \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

$$C\gamma^0 U^*(p, 1) = i\gamma^2 \sqrt{E+m} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{1}{E+m} \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}^* =$$

$$= \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ p_3/(E+m) \\ (p_1 - ip_2)/(E+m) \end{pmatrix} \sqrt{E+m} = \begin{pmatrix} (p_1 - ip_2)/(E+m) \\ -p_3/(E+m) \\ 0 \\ 1 \end{pmatrix} \sqrt{E+m} \quad ✓$$

$$V(p, 1) = \sqrt{E+m} \begin{pmatrix} \frac{1}{E+m} \begin{pmatrix} p_1 - ip_2 \\ -p_3 \end{pmatrix} \\ \vdots \end{pmatrix} = \sqrt{E+m} \begin{pmatrix} (p_1 - ip_2)/(E+m) \\ -p_3/(E+m) \\ \vdots \end{pmatrix} ✓$$

2. $(i\not\nabla + q\not A - m)\Psi = 0$

or $(i\gamma^\mu \nabla_\mu + q\gamma^\mu A_\mu - m)\Psi = 0$

$* \rightarrow \left(-i(\gamma^\mu)^* \nabla_\mu + q(\gamma^\mu)^* A_\mu - m\right)\Psi^* = 0$

$\times\ C\gamma^0$ on left $\rightarrow \left(-iC\gamma^0(\gamma^\mu)^* \nabla_\mu + qC\gamma^0(\gamma^\mu)^* A_\mu - mC\gamma^0\right)\Psi^* = 0$

$C\gamma^0(\gamma^\mu)^* = -\gamma^\mu C\gamma^0 \Rightarrow \left(i\gamma^\mu C\gamma^0 \nabla_\mu - q\gamma^\mu C\gamma^0 A_\mu - mC\gamma^0\right)\Psi^* = 0$

$\Rightarrow (i\gamma^\mu \nabla_\mu - q\gamma^\mu A_\mu - m)\underbrace{C\gamma^0\Psi^*}_{\Psi_c} = 0$

$\Rightarrow \underline{(i\not\nabla - q\not A - m)\Psi_c = 0}$

3) $$i\frac{\partial\phi}{\partial t} = \left(\frac{(\hat{\vec{p}} + q\vec{A})^2}{2m} + \frac{q}{2m}\,\vec{\sigma}\cdot\vec{B} - q A^0 \right)\phi$$

$$\hat{\vec{S}} = \frac{1}{2}\vec{\sigma}$$

$$(\hat{\vec{p}} + q\vec{A})^2\phi =$$

$$= \sum_i (\hat{p}_i + qA_i)(\hat{p}_i + qA_i)\phi$$

$$= \sum_i \left(-i\frac{\partial}{\partial x_i} + \frac{q}{2}\epsilon_{ijk}B_j x_k\right)\left(-i\frac{\partial}{\partial x_i} + \frac{q}{2}\epsilon_{ilm}B_l x_m\right)\phi$$

$$= (\hat{\vec{p}})^2\phi + \frac{q}{2}\sum_i \left[-i\frac{\partial}{\partial x_i}\left(\epsilon_{ilm}B_l x_m \phi\right) + \epsilon_{ijk}B_j x_k\left(-i\frac{\partial}{\partial x_i}\phi\right)\right] + \mathcal{O}(\vec{A}^2)$$

$$\cong (\hat{\vec{p}})^2\phi - i\frac{q}{2}\sum_i \left[\cancel{\epsilon_{ilm}B_l\delta_{im}\phi} + \epsilon_{ilm}B_l x_m\frac{\partial\phi}{\partial x_i} + \epsilon_{ijk}B_j x_k\frac{\partial\phi}{\partial x_i}\right]$$

$$= (\hat{\vec{p}})^2\phi - iq\,\vec{B}\cdot(\vec{x}\times\vec{\nabla}\phi)$$

$$= (\hat{\vec{p}})^2\phi + q\,\vec{B}\cdot\hat{\vec{L}}\phi$$

$$\vec{\sigma} \equiv 2\hat{\vec{S}}$$

$$\Rightarrow\ i\frac{\partial\phi}{\partial t} = \left(\frac{(\hat{\vec{p}})^2}{2m} + \frac{q}{2m}(\hat{\vec{L}} + 2\hat{\vec{S}})\cdot\vec{B} - qA^0\right)\phi + \mathcal{O}(\vec{A}^2)$$