RQM HW 4 (2006)

SOLUTIONS

1)
$$U(p,s) = \sqrt{E+m}\begin{pmatrix} \phi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\phi_s \end{pmatrix} \qquad \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

since $\vec{\sigma}$ is hermitian, and $\vec{p}$ and ϕ_s are real

$$U^+ = \sqrt{E+m}\begin{pmatrix} \phi_s^T & \phi_s^T\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}$$

$$U^+U = (E+m)\left(\phi_s^T\phi_s + \phi_s^T\,\frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2}\,\phi_s\right)$$

$$\vec{\sigma}\cdot\vec{p} = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{p})^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$= \begin{pmatrix} \vec{p}^{\,2} & 0 \\ 0 & \vec{p}^{\,2} \end{pmatrix} = \vec{p}^{\,2}\,\mathbb{1}_2 = (E^2 - m^2)\,\mathbb{1}_2$$

so $U^+U = (E+m)\left(1 + \frac{E^2-m^2}{(E+m)^2}\right) = E+m + E-m = \underline{\underline{2E}}$

$$\bar{U} = U^\dagger \gamma^0 = \sqrt{E+m}\left(\phi_s^T \quad \phi_s^T \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad (2)$$

$$= \sqrt{E+m}\left(\phi_s^T \quad -\phi_s^T \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)$$

$$\bar{U}U = (E+m)\left(1 - \frac{E^2-m^2}{(E+m)^2}\right) = E+m-(E-m) = \underline{\underline{2m}}$$

2) Invariance of Dirac Equ. under time reversal $\mathcal{T}$

$$\psi(x) \to \psi'(x') = T\psi^*(x) = -\gamma^1\gamma^3\psi^*(x)$$

$$(i\not{\partial}-m)\psi(x) = (i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2 + i\gamma^3\partial_3 - m)\psi(x) = 0$$

Complex conjugate (note $\gamma^0 = (\gamma^0)^*$, $\gamma^1 = (\gamma^1)^*$, $\gamma^3 = (\gamma^3)^*$, $\gamma^2 = -(\gamma^2)^*$)

$$\Rightarrow (-i\gamma^0\partial_0 - i\gamma^1\partial_1 + i\gamma^2\partial_2 - i\gamma^3\partial_3 - m)\psi^*(x) = 0$$

Multiply with $(-\gamma^1\gamma^3)$

$$(i\gamma^1\gamma^3\gamma^0\partial_0 + i\gamma^1\gamma^3\gamma^1\partial_1 - i\gamma^1\gamma^3\gamma^2\partial_2 + i\gamma^1\gamma^3\gamma^3\partial_3 - m(-\gamma^1\gamma^3))\psi^*(x) = 0$$

$$\Rightarrow [-i\gamma^0(-\gamma^1\gamma^3)\partial_0 + i\gamma^1(-\gamma^1\gamma^3)\partial_1 + i\gamma^2(-\gamma^1\gamma^3)\partial_2 + i\gamma^3(-\gamma^1\gamma^3)\partial_3$$
$$- m(-\gamma^1\gamma^3)]\psi^*(x) = 0$$

$$\partial_0' = -\partial_0, \quad \partial_i' = \partial_i$$

$$\Rightarrow [i\gamma^0\partial_0' + i\vec{\gamma}\cdot\vec{\nabla}' - m](-\gamma^1\gamma^3)\psi^*(x) = 0$$

$$\Rightarrow \underline{\underline{(i\not{\partial}' - m)\psi'(x') = 0}}$$

3) $\mathcal{P}$: $\Psi \to \Psi' = \gamma^0 \Psi$ $\qquad t \to t' = t\,,\ \vec{x} \to \vec{x}' = -\vec{x}$

$$\mathcal{P}: \ \bar{\Psi}\gamma_5\Psi = \Psi^\dagger\gamma^0\gamma_5\Psi$$

$$\xrightarrow{\mathcal{P}} (\gamma^0\Psi)^\dagger\gamma^0\gamma_5\gamma^0\Psi$$

$$= \Psi^\dagger \underbrace{\underbrace{(\gamma^0)^\dagger}_{=\gamma^0}\gamma^0}_{\gamma^0\gamma^0 = \mathbb{1}}\gamma_5\gamma^0\Psi$$

$$= \Psi^\dagger\gamma_5\gamma^0\Psi$$

$$= -\Psi^\dagger\gamma^0\gamma_5\Psi = \underline{\underline{-\bar{\Psi}\gamma_5\Psi}}$$

$$\mathcal{C}: \ \Psi \to C\gamma^0\Psi^* \qquad C = i\gamma^2\gamma^0$$
$$\Psi \to i\gamma^2\Psi^*$$

$$\bar{\Psi}\gamma^\mu\gamma_5\Psi = \Psi^\dagger\gamma^0\gamma^\mu\gamma_5\Psi \qquad \text{remember } \Psi^\dagger = (\Psi^*)^T$$

$$\xrightarrow{\mathcal{C}} (i\gamma^2\Psi^*)^\dagger\gamma^0\gamma^\mu\gamma_5 i\gamma^2\Psi^*$$

$$= \Psi^T(\gamma^2)^\dagger\cancel{(-i)}\gamma^0\gamma^\mu\gamma_5\cancel{i}\gamma^2\Psi^* \qquad (\gamma^2)^\dagger = -\gamma^2$$

$$= -\Psi^T\gamma^2\gamma^0\gamma^\mu\gamma_5\gamma^2\Psi^* \qquad \begin{aligned}\gamma^2\gamma^0 &= -\gamma^0\gamma^2\\ \gamma_5\gamma^2 &= -\gamma^2\gamma_5\end{aligned}$$

$$= -\Psi^T\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma_5\Psi^*$$

$$\mu \neq 2 \qquad = -\Psi^T\gamma^0\gamma^\mu\gamma_5\Psi^*$$
$$\mu = 2 \qquad = \Psi^T\gamma^0\gamma^2\gamma_5\Psi^*$$

Now $(\Psi^T \Gamma \Psi^*) = (\,\cdot\,)^T = (\Psi^\dagger \Gamma^T \Psi)$ (4)

$$\mu \neq 2 \qquad = -\Psi^\dagger \gamma_5^T (\gamma^\mu)^T (\gamma^0)^T \Psi$$
$$\mu = 2 \qquad = \Psi^\dagger \gamma_5^T (\gamma^2)^T (\gamma^0)^T \Psi$$

$\gamma^0, \gamma^1, \gamma^3$ are real, so $(\gamma^0)^T = (\gamma^0)^\dagger = \gamma^0$

$(\gamma^1)^T = (\gamma^1)^\dagger = -\gamma^1$

$(\gamma^3)^T = (\gamma^3)^\dagger = -\gamma^3$

γ^2 is imaginary, so $(\gamma^2)^T = -(\gamma^2)^\dagger = \gamma^2$

and $\gamma_5 = \gamma_5^T$

$$\mu = 0 \qquad = -\Psi^\dagger \gamma_5 \gamma^0 \gamma^0 \Psi = -\Psi^\dagger \gamma^0 \gamma^0 \gamma_5 \Psi = -\bar{\Psi} \gamma^0 \gamma_5 \Psi$$

$$\mu = 1 \qquad = +\Psi^\dagger \gamma_5 \gamma^1 \gamma^0 \Psi = -\Psi^\dagger \gamma^0 \gamma^1 \gamma_5 \Psi = -\bar{\Psi} \gamma^1 \gamma_5 \Psi$$

$$\mu = 2 \qquad = \Psi^\dagger \gamma_5 \gamma^2 \gamma^0 \Psi = -\Psi^\dagger \gamma^0 \gamma^2 \gamma_5 \Psi = -\bar{\Psi} \gamma^2 \gamma_5 \Psi$$

$$\mu = 3 \qquad = \Psi^\dagger \gamma_5 \gamma^3 \gamma^0 \Psi = -\Psi^\dagger \gamma^0 \gamma^3 \gamma_5 \Psi = -\bar{\Psi} \gamma^3 \gamma_5 \Psi$$

$$= \underline{\underline{-\bar{\Psi} \gamma^\mu \gamma_5 \Psi}}$$

4) $$U(p,s) = \sqrt{E+m}\begin{pmatrix} \phi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi_s \end{pmatrix}$$

$$\vec{p}=0 : E=m \Rightarrow U(0,s) = \sqrt{2m}\begin{pmatrix} \phi_s \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$S = e^{-i\omega\sigma(K_z)} = e^{-\frac{i\omega}{2}iM}, \quad \sigma = iM, \quad M = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$= \mathbb{1}\sum_{n=0}^{\infty}\frac{1}{2n!}\left(+\frac{\omega}{2}\right)^{2n} \qquad M^2 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = M^{2n}$$

$$+M\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\omega}{2}\right)^{2n+1} \qquad M^{2n+1} = M$$

$$= \begin{pmatrix} \cosh\frac{\omega}{2}\,\mathbb{1}_2 & \sinh\frac{\omega}{2}\,\sigma_z \\ \sinh\frac{\omega}{2}\,\sigma_z & \cosh\frac{\omega}{2}\,\mathbb{1}_2 \end{pmatrix}$$

Now

$$SU(0s) = \sqrt{2m}\begin{pmatrix} \cosh\frac{\omega}{2}\,\phi_s \\ \sinh\frac{\omega}{2}\,\sigma_z\phi_s \end{pmatrix} \equiv U(\vec{p}=(0,0,p_z),s)$$

$$= \sqrt{E+m}\begin{pmatrix} \phi_s \\ \frac{p_z\sigma_z}{E+m}\phi_s \end{pmatrix}$$

$$\Rightarrow \sqrt{2m}\cosh\frac{\omega}{2} = \sqrt{E+m}$$

$$\sqrt{2m}\sinh\frac{\omega}{2} = \sqrt{E+m}\,\frac{p_z}{E+m}$$

$$\Rightarrow \cosh\frac{\omega}{2} = \sqrt{\frac{E+m}{2m}}$$

$$\sinh\frac{\omega}{2} = \frac{p_z}{\sqrt{(2m)(E+m)}}$$