$$H = c\,\underline{\alpha}\cdot\underline{p} + \beta mc^2 = c\,\alpha_i p_i + \beta mc^2$$

$$\underline{L} = \underline{r}\times\underline{p} = \epsilon_{ijk} r_j p_k$$

$$[H, L_i] = c[\alpha_l p_l, \epsilon_{ijk} r_j p_k] + mc^2[\beta, \epsilon_{ijk} r_j p_k]$$

$$= c\,\epsilon_{ijk}[\alpha_l p_l, r_j p_k] + mc^2[\beta, r_j p_k]\,\epsilon_{ijk}$$

$$= \epsilon_{ijk}\Big(c\,\alpha_l[p_l, r_j p_k] + c[\alpha_l, r_j p_k]p_l + mc^2 r_j[\beta, p_k] + mc^2[\beta, r_j]p_k \Big)$$

$$= c\,\epsilon_{ijk}\Big(\alpha_l r_j \cancel{[p_l, p_k]} + \alpha_l[p_l, r_j]p_k + \cancel{c}\, r_j \cancel{[\alpha_l, p_k]} p_l + \cancel{c}\cancel{[\alpha_l, r_j]} p_k p_l + mc\, r_j \cancel{[\beta, p_k]} + mc \cancel{[\beta, r_j]} p_k \Big)$$

$$[p_l, p_k] = 0 \qquad [p_l, r_j] = -i\hbar[\partial_l, r_j]$$

$$= -i\hbar\left(\delta_{lj} + \cancel{r_j\partial_l} - \cancel{r_j\partial_l}\right)$$

$$= -i\hbar\,\delta_{lj}$$

In this context, $p_k \equiv p_k \mathbb{1}_4$; $r_k \equiv r_k \mathbb{1}_4$

$$\Rightarrow [\alpha_l, p_k] = 0$$
$$[\alpha_l, r_j] = 0$$
$$[\beta, p_k] = 0$$
$$[\beta, r_j] = 0$$

$$\Rightarrow [H, L_i] = c\,\epsilon_{ijk}\left(-i\hbar\,\delta_{lj}\,\alpha_l p_k\right)$$

$$= -i\hbar c\, \epsilon_{ijk}\alpha_j p_k$$

$$\therefore\ \underline{\underline{[H, \underline{L}] = -i\hbar c(\underline{\alpha} \times \underline{p})}}$$

$$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$[H, \Sigma^i] = c[\alpha_j p_j \mathbb{1}_4, \Sigma^i] + mc^2[\beta, \Sigma^i]$$

$$= c p^j [\alpha^j, \Sigma^i] + mc^2[\beta, \Sigma^i]$$

$$\alpha^j \Sigma^i = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma^j\sigma^i \\ \sigma^j\sigma^i & 0 \end{pmatrix}$$

$$\Sigma^i \alpha^j = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma^i\sigma^j \\ \sigma^i\sigma^j & 0 \end{pmatrix}$$

$$\beta \Sigma^i = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$\Sigma^i \beta = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$\therefore \ [\alpha^j, \Sigma^i] = \begin{pmatrix} 0 & \sigma^j\sigma^i - \sigma^i\sigma^j \\ \sigma^j\sigma^i - \sigma^i\sigma^j & 0 \end{pmatrix}$$

$$[\beta, \Sigma^i] = 0$$

$$\therefore \ [H, \Sigma^i] = c \begin{pmatrix} 0 & (\underline{\sigma}\cdot\underline{p})\sigma^i - \sigma^i(\underline{\sigma}\cdot\underline{p}) \\ (\underline{\sigma}\cdot\underline{p})\sigma^i - \sigma^i(\underline{\sigma}\cdot\underline{p}) & 0 \end{pmatrix}$$

· Algebra of σ's: $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$

$$(\underline{\sigma}\cdot\underline{p})\sigma^i - \sigma^i(\underline{\sigma}\cdot\underline{p})$$

$$= p^j(\sigma^j\sigma^i - \sigma^i\sigma^j)$$

$$= (\cancel{\delta^{ji}} + i\epsilon^{jik}\sigma^k - \cancel{\delta^{ij}} - i\epsilon^{ijk}\sigma^k)p^j$$

$$= -2i\epsilon^{ijk}\sigma^k p^j = 2i\epsilon^{ikj}\sigma^k p^j = \underline{\underline{2i(\underline{\sigma}\times\underline{p})^i}}$$

$$\therefore [H, \tfrac{\hbar}{2}\Sigma^i] = i\hbar c \begin{pmatrix} 0 & \epsilon^{ijk}\sigma^j p^k \\ \epsilon^{ijk}\sigma^j p^k & 0 \end{pmatrix}$$

$$= i\hbar c\, \epsilon^{ijk} \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} p^k$$

$$= i\hbar c\, \epsilon^{ijk} \alpha^j p^k$$

$$\therefore [H, \tfrac{\hbar}{2}\underline{\Sigma}] = i\hbar c(\underline{\alpha} \times \underline{p})$$