RQM 3

$$\boxed{\begin{aligned}&(\alpha^i)^2 = \mathbb{1}_4 \ ; \ \beta^2 = \mathbb{1}_4\\ &\alpha^i\alpha^j + \alpha^j\alpha^i = 0\\ &\alpha^i\beta + \beta\alpha^i = 0\end{aligned}}$$

1. $\gamma^\mu : \gamma^0 = \beta \ ; \ \gamma^i = \beta\alpha^i$

$$\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = 2\beta^2 = 2\mathbb{1}_4$$

$$\{\gamma^0, \gamma^i\} = \{\beta, \beta\alpha^i\} = \beta^2\alpha^i + \beta\alpha^i\beta$$
$$= \beta^2\alpha^i - \beta^2\alpha^i = 0$$

$$\underbrace{\{\gamma^i, \gamma^i\}}_{\text{No sum}} = \{\beta\alpha^i, \beta\alpha^i\} = \beta\alpha^i\beta\alpha^i + \beta\alpha^i\beta\alpha^i$$
$$= -(\beta^2(\alpha^i)^2 + \beta^2(\alpha^i)^2)$$
$$= -2\mathbb{1}_4\mathbb{1}_4 = -2\mathbb{1}_4$$

$$\boxed{\begin{aligned}\{\gamma^i, \gamma^j\} &= \{\beta\alpha^i, \beta\alpha^j\}\\ &= -\beta^2\alpha^i\alpha^j - \beta^2\alpha^j\alpha^i\\ &= -\beta^2(\alpha^i\alpha^j + \alpha^j\alpha^i)\\ &= \underline{\underline{0}}\ .\end{aligned}}$$

As $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

$$\Rightarrow \underline{\underline{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4}}$$

2. $\cancel{(\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0 = \beta^2\beta = \beta = \gamma^0}$

$$\boxed{\begin{aligned}&\gamma^0 = \beta^\dagger = \beta\\ &\gamma^i = (\beta\alpha^i) \ ; (\alpha^i)^\dagger = +\alpha^i\end{aligned}}$$

$\cancel{(\gamma^i)^\dagger =}$

$$(\gamma^0)^\dagger = \beta^\dagger = \beta = \beta^2\beta = \beta\beta\beta = \gamma^0\gamma^0\gamma^0$$

$$(\gamma^i)^\dagger = (\beta\alpha^i)^\dagger = (\alpha^i)^\dagger\beta^\dagger = \alpha^i\beta = \beta^2\alpha^i\beta = \beta(\beta\alpha^i)\beta$$
$$= \gamma^0\gamma^i\gamma^0$$

$$\therefore \quad \underline{\underline{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= i\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & \sigma^1\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & -\sigma^1\sigma^2\sigma^3 \\ -\sigma^1\sigma^2\sigma^3 & 0 \end{pmatrix}$$

$$\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$\therefore \gamma_5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

$$\begin{aligned}
\gamma^0 \gamma_5 \gamma^0 &= i(\gamma^0)^2 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \\
&= i \gamma^1 \gamma^2 (-\gamma^0 \gamma^3) \\
&= i \gamma^1 \gamma^0 \gamma^2 \gamma^3 \\
&= -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\
&= -\gamma_5
\end{aligned}$$

$$\begin{aligned}
(\gamma_5)^\dagger &= -i (\gamma^3)^\dagger (\gamma^2)^\dagger (\gamma^1)^\dagger (\gamma^0)^\dagger \\
&= +i \gamma^3 \gamma^2 \gamma^1 \gamma^0 \\
&= -i \gamma^0 \gamma^3 \gamma^2 \gamma^1 \\
&= -i \gamma^0 \gamma^1 \gamma^3 \gamma^2 \\
&= i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\
&= \gamma_5
\end{aligned}$$

$$\boxed{\begin{aligned} (\gamma^1)^\dagger &= -\gamma^1 \\ (\gamma^0)^\dagger &= \gamma^0 \end{aligned}}$$

$$\therefore \gamma^0 \gamma_5 \gamma^0 = -\gamma_5 = -(\gamma_5)^\dagger$$

$$\cancel{\therefore \gamma^0(\gamma_5\gamma^\mu)\gamma^0 = (\gamma_5\gamma^\mu)^\dagger}$$

$$\gamma^0(\gamma_5\gamma^\mu)\gamma^0 = i(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^0$$
$$= i\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^0 \qquad (\dagger)$$

If $\gamma^\mu = \gamma^0$ $\quad (\dagger) = i\gamma^1\gamma^2\gamma^3\gamma^0\gamma^0$

$$= -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0$$
$$= -\gamma^5\gamma^0$$
$$= -i\gamma^0\gamma^2\gamma^3\gamma^1\gamma^0$$
$$= +i\gamma^0\gamma^3\gamma^2\gamma^1\gamma^0$$
$$= (\gamma^0)^\dagger(+i)^\dagger(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger$$
$$= (\gamma^0)^\dagger(\gamma_5)^\dagger = \underline{\underline{(\gamma_5\gamma^0)^\dagger}}$$

If $\gamma^\mu = \gamma^i$ $\quad (\dagger) = i\gamma^1\gamma^2\gamma^3\gamma^i\gamma^0$

$$\cancel{= -i\gamma^i\gamma^1\gamma^2\gamma^3\gamma^0}$$
$$\cancel{= i\gamma^i\gamma^3\gamma^2\gamma^1\gamma^0}$$

Can show $\{\gamma^i, \gamma_5\} = 0$

$\therefore \quad (\dagger) = -\gamma^0\gamma^i\gamma_5\gamma^0$

$$= +i\gamma^i\gamma^1\gamma^2\gamma^3\gamma^0$$
$$= -i\gamma^i\gamma^3\gamma^2\gamma^1\gamma^0$$
$$= (\gamma^i)^\dagger(i\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger$$
$$= (\gamma^i)^\dagger(\gamma_5)^\dagger = \underline{\underline{(\gamma_5\gamma^i)^\dagger}}$$

$$\therefore \quad \underline{\underline{\gamma^0(\gamma_5\gamma^\mu)\gamma^0 = (\gamma_5\gamma^\mu)^\dagger}}$$

<u>Alternative:</u> $\quad \{\gamma^\mu, \gamma_5\} = 0$

$$\Rightarrow \gamma^0\gamma_5\gamma^\mu\gamma^0 = -\gamma_5(\gamma^0\gamma^\mu\gamma^0)$$
$$= -\gamma_5(\gamma^\mu)^\dagger$$

$$\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu \Rightarrow \cancel{(\gamma^\mu\gamma_5)^\dagger} \; (\gamma_5)(\gamma^\mu)^\dagger = -(\gamma^\mu)^\dagger\gamma_5$$
$$\Rightarrow \{(\gamma^\mu)^\dagger, \gamma_5\} = 0$$

$$\therefore \gamma^0\gamma_5\gamma^\mu\gamma^0 = +(\gamma^\mu)^\dagger\gamma_5$$
$$= (\gamma^\mu)^\dagger\gamma_5^\dagger = \underline{\underline{(\gamma_5\gamma^\mu)^\dagger}}$$