Homework Week 01 Solutions

1) $$L_x = -i\hbar(y\partial_z - z\partial_y)$$
$$L_y = -i\hbar(z\partial_x - x\partial_z)$$
$$L_z = -i\hbar(x\partial_y - y\partial_x)$$

$$[L_x, L_y]\psi = -\hbar^2\Big\{\underline{[y\partial_z, z\partial_x]} - \cancel{[y\partial_z, x\partial_z]} - \cancel{[z\partial_y, z\partial_x]}$$
$$+ \underline{[z\partial_y, x\partial_z]}\Big\}\psi$$
$$= -\hbar^2\Big\{y(\partial_x + \cancel{z\partial_x\partial_z}) - \cancel{zy\partial_x\partial_z} + \cancel{zx\partial_y\partial_z}$$
$$- x(\partial_y + \cancel{z\partial_y\partial_z})\Big\}\psi =$$
$$= (i\hbar)(i\hbar)(-)\{x\partial_y - y\partial_x\}\psi = i\hbar L_z$$

+ cyclic perms for other

$$[L_x, \vec\nabla^2]\psi = \cancel{[L_x, \partial_x^2]\psi} + [L_x, \partial_y^2]\psi + [L_x, \partial_z^2]\psi =$$
$$= (-i\hbar)\Big\{[y\partial_z, \partial_y^2] - \cancel{[z\partial_y, \partial_y^2]} + \cancel{[y\partial_z, \partial_z^2]} - [z\partial_y, \partial_z^2]\Big\}\psi$$
$$= (-i\hbar)\Big\{y\partial_z\partial_y^2 - \partial_y(\cancel{\partial_z} + y\partial_y\partial_z) - z\partial_y\partial_z^2$$
$$+ \partial_z(\cancel{\partial_y} + z\partial_y\partial_z)\Big\}\psi$$
$$= (-i\hbar)\Big\{\cancel{y\partial_z\partial_y^2} - \cancel{\partial_y\partial_z} - \cancel{y\partial_y^2\partial_z} - \cancel{z\partial_y\partial_z^2} + \cancel{\partial_y\partial_z} + \cancel{z\partial_y\partial_z^2}\Big\}\psi$$
$$= 0 \quad \checkmark$$

+ cyclic perms $\Rightarrow [\vec L, \vec\nabla^2]\psi = 0$

$$[L_x, \vec{r}^2]\psi = \cancel{[L_x, x^2]}\psi + [L_x, y^2]\psi + [L_x, z^2]\psi$$

$$= -i\hbar\left\{\cancel{[y\partial_z, y^2]} - [z\partial_y, y^2] + [y\partial_z, z^2] - \cancel{[z\partial_y, z^2]}\right\}\psi$$

$$= -i\hbar\left\{-z\partial_y(y^2\psi) + y^2 z\partial_y\psi + y\partial_z(z^2\psi) - z^2 y\partial_z\psi\right\} =$$

$$= -i\hbar\left\{-\cancel{2zy\psi} - \cancel{zy^2\partial_y\psi} + \cancel{y^2 z\partial_y\psi} + \cancel{yz^2\partial_z\psi} + \cancel{2yz\psi} - \cancel{z^2 y\partial_z\psi}\right\} = 0 \checkmark$$

QM Hamiltonians that are functions of $\vec{\nabla}^2$ and $r = \sqrt{r^2}$ alone lead to conserved AM.

2) $\vec{J} = \vec{L} + \vec{S}$ $\qquad [L_i, S_j] = 0$

$A = \vec{L} \cdot \vec{S}$ (proportional to Spin-Orbit coupling)

compute: $[\vec{L}, A]$

$[\vec{S}, A]$

$\Rightarrow [\vec{J}, A] = 0$

Consequence?

3) $[L_x, L_y] = i\hbar L_z \,, \dots \qquad [L_i, S_j] = 0$

$[S_x, S_y] = i\hbar S_z \,, \dots$

$[\vec{L}, A]$:

$$[L_x, \cancel{L_x S_x} + L_y S_y + L_z S_z] =$$

$$= S_y [L_x, L_y] + S_z [L_x, L_z] =$$

$$= i\hbar S_y L_z - i\hbar S_z L_y$$

$$[L_y, A] = i\hbar (S_z L_x - S_x L_z)$$

$$[L_z, A] = i\hbar (S_x L_y - S_y L_x)$$

$[\vec{S}, A]$:

$$[S_x, \cancel{S_x L_x} + S_y L_y + S_z L_z] =$$

$$= L_y [S_x, S_y] + L_z [S_x, S_z] =$$

$$= i\hbar (L_y S_z - L_z S_y)$$

$$[S_y, A] = i\hbar (L_z S_x - L_x S_z)$$

$$[S_z, A] = i\hbar (L_x S_y - L_y S_x)$$

$$\Rightarrow [S_x, A] = -[L_x, A] \,, \dots$$

$$\Rightarrow [J_x, A] = 0 \,, \dots$$

$$\Rightarrow [\vec{J}, A] = 0$$

- Adding the relativistic Spin-Orbit coupling prop. to A to $\hat{H}$: $\Rightarrow$ $\vec{L}$ not conserved
  $\vec{J}$ IS conserved quantity

3) $H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + cz$

which components of $\vec{P}$ and $\vec{L}$ operators correspond to conserved quantities?

since $\vec{\nabla}$ commutes with $\vec{P}$ and $\vec{L}$, we only need to check which components commute with z.

$$[L_x, z]\psi = -i\hbar[(y\partial_z - z\partial_y)(z\psi) - z(y\partial_z - z\partial_y)\psi] =$$

$$= -i\hbar y\psi \neq 0$$

$$[L_y, z]\psi = -i\hbar[(z\partial_x - x\partial_z)(z\psi) - z(z\partial_x - x\partial_z)\psi]$$

$$= +i\hbar x\psi \neq 0$$

$$[L_z, z]\psi = -i\hbar[x\partial_y - y\partial_x, z]\psi = 0$$

L_z is conserved quantity

$$\vec{P} = -i\hbar\vec{\nabla} = -i\hbar(\partial_x, \partial_y, \partial_z)$$

$$[P_x, z]\psi = 0$$

$$[P_y, z]\psi = 0$$

$$[P_z, z]\psi = -i\hbar[\partial_z(z\psi) - z\partial_z\psi] = -i\hbar\psi \neq 0$$

P_x and P_y are conserved