

MSci EXAMINATION

PHY-415 (MSci 4242) Relativistic Waves and Quantum Fields

Time Allowed: 2 hours 30 minutes

Date: XXth May, 2010

Time: 14:30 - 17:00

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [] after each part of a question. Course work comprises 10% of the final mark.**

Throughout the paper units are used such that $\hbar = c = 1$ (Natural Units).
A FORMULA SHEET is provided at the end of the questions paper.

Numeric calculators are not permitted in this examination. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A. Brandhuber (MO), Dr. R. Russo (DMO)

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QUESTION 1: The Klein-Gordon and Dirac equations

- (a) Starting from the relativistic energy-momentum relation give a derivation of the Klein-Gordon (KG) equation for the wave function $\Psi(x^\mu)$. Using this result show that the quantities $\rho_{\text{KG}} = \frac{i}{2m} (\Psi^* \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^*}{\partial t} \Psi)$ and $\vec{j}_{\text{KG}} = -\frac{i}{2m} (\Psi^* \vec{\nabla} \Psi - \vec{\nabla} \Psi^* \Psi)$ obey a continuity equation. Prove that the KG equation is covariant, i.e. show that if $\Psi(x^\mu)$ obeys the KG equation then also the Lorentz transformed wave function $\Psi(\Lambda^\mu{}_\nu x^\nu)$ obeys the same equation. [7]
- (b) Find positive and negative energy plane wave solutions of the KG equation and express the most general solution to the KG equation in terms of these. For both types of plane wave solutions calculate the density ρ_{KG} . Using this result explain why Schrödinger gave up on the KG equation and instead introduced his non-relativistic wave equation. Which property of the KG equation is the reason for this problem. Describe briefly what radically different interpretations are given to Ψ and ρ_{KG} in Quantum Field Theory. [6]
- (c) Describe how Dirac proposed to circumvent the problems of the KG equation and, hence, motivate the form of the ansatz for the Dirac equation. Give a derivation of the Dirac equation stating clearly the novel features of the Dirac equation and its wavefunction. Derive the continuity equation associated with the Dirac equation and hence argue that the probability density is positive and conserved. [7]

QUESTION 2: Solutions of the Dirac equation

- (a) Consider positive energy plane wave solutions of the Dirac equation (using the Dirac matrices given on the FORMULA SHEET)

$$\Psi = \mathcal{N} e^{-ip \cdot x} U(p, s) \quad \text{with} \quad U(p, s) = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where ϕ and χ denote two component column spinors. Derive equations for ϕ and χ and hence find the general expression for $U(p, s)$ up to a normalisation factor \mathcal{N} . [6]

- (b) Using the positive energy plane wave solutions $U(p, s)$ constructed above, find the normalisation constant \mathcal{N} by requiring $U^\dagger(p, s)U(p, s) = 2E$. Hence show that $\bar{U}(p, s)U(p, s) = 2m$. Note: you may use the identity $(\vec{\sigma} \cdot \vec{p})^2 = p^2 \mathbb{1}_2$ without proof. [4]
- (c) Find the constant parts of the plane wave solutions $U(p, s)$ with the correct normalisation constants for the case of a Dirac particle at rest $\vec{p} = 0$ and a Dirac particle moving in the z direction $\vec{p} = (0, 0, p_z)$. Show that the two U 's are related by a boost in the z -direction where the boost of a Dirac spinor in the z -direction is described by the matrix

$$S = e^{-i\omega\sigma(K_z)}, \quad \text{where} \quad \sigma(K_z) = \frac{i}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Show that the correct boost parameter ω obeys $\cosh \frac{\omega}{2} = \sqrt{\frac{E+m}{2m}}$ and $\sinh \frac{\omega}{2} = \frac{p_z}{\sqrt{2m(E+m)}}$.

Remember, that E , p_z and m are related as $E^2 = p_z^2 + m^2$ in this example. [10]

QUESTION 3: Symmetries and gauge fields

(a) State what is meant by covariance (form invariance) of a relativistic wave equation under symmetry transformations. Symmetries of relativistic wave equations (and quantum field theories) can be continuous global, continuous local or discrete; give one example of each type. [4]

(b) Show that the electromagnetic field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under the gauge transformation $A^\mu \rightarrow A^\mu + \partial^\mu \chi$, with χ an arbitrary, real function of the space-time coordinates. How must the Dirac field Ψ transform under a gauge transformation, in order that the combined transformation of A^μ and Ψ preserves the QED Lagrangian (proof required)

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}(i\cancel{\partial} + q\cancel{A} - m)\Psi . \quad [6]$$

(c) The Lagrangian density for a massive vector field B_μ is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 B_\mu B^\mu ,$$

where the fieldstrength is defined as $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and m is the mass of the vector field. Find the Euler-Lagrange equations of motion for B_ν and show (for $m \neq 0$) that they imply

$$\partial_\nu B^\nu = 0 . \quad [7]$$

Is the Lagrangian density \mathcal{L} invariant under gauge transformations $B_\nu \rightarrow B_\nu + \partial_\nu \chi(x)$, where $\chi(x)$ is an arbitrary function depending on x^ν ? Justify your answer. [1]

Find plane wave solutions of the form $B^\mu = \epsilon^\mu e^{-ip \cdot x}$ and show that the polarisation vector is transverse i.e. $\epsilon \cdot p = 0$. [2]

QUESTION 4: The neutral Klein-Gordon field

- (a) The free, neutral Klein-Gordon field $\phi = \phi^\dagger$ has Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2).$$

Obtain the field equation for ϕ and the Hamiltonian density \mathcal{H} in terms of ϕ and its derivatives. [4]

- (b) The free, neutral Klein Gordon field may be expanded in the form

$$\phi = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right],$$

with $E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}$ and $k \cdot x = E_{\vec{k}}t - \vec{k} \cdot \vec{x}$. Show that the commutation relations of the operators a and a^\dagger (see FORMULA SHEET at the end of exam paper) are compatible with the equal time commutation relation

$$[\phi(t, \vec{x}), \Pi(t, \vec{x}')] = i\delta^{(3)}(\vec{x} - \vec{x}'),$$

where $\Pi = \dot{\phi}$ denotes the momentum canonically conjugate to ϕ . [6]

- (c) Using the commutation relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$ given on the FORMULA SHEET calculate the commutator of two field operators at general space-time points x and y

$$i\Delta(x - y) = [\phi(x), \phi(y)].$$

Note that you do not have to perform the final three-momentum integral explicitly. Show that the result is Lorentz invariant and vanishes for space like separations $(x - y)^2 < 0$. Discuss the physical significance of the latter property of $[\phi(x), \phi(y)]$. [5]

- (d) The normal ordered form of the Hamiltonian operator of the real Klein-Gordon field can be written in the form

$$H = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} E_{\vec{k}} a^\dagger(\vec{k})a(\vec{k}).$$

Show that the vacuum expectation value of this Hamiltonian, i.e. the vacuum energy $\langle 0|H|0\rangle$ vanishes giving the appropriate definition of the vacuum state $|0\rangle$.

Show that $[H, a^\dagger(\vec{p})] = E_{\vec{p}}a^\dagger(\vec{p})$ using the commutation relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$ given on the FORMULA SHEET. Hence, what is the physical interpretation of the operators $a^\dagger(\vec{p})$? [5]

QUESTION 5: The S -matrix

Consider the theory of an interacting neutral scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 .$$

From Dyson's formula one infers that the S -matrix is given by

$$S = T \exp \left(i \int d^4x : \mathcal{L}_{\text{int}}(x) : \right) .$$

- (a) Define the free Lagrangian \mathcal{L}_0 and the interaction part \mathcal{L}_{int} . Write down the first three terms in a perturbative expansion of S . Briefly explain the meaning of the symbol T . What is the meaning of the normal ordering prescription “: :”? Hence what are $: a(\vec{k}) a^\dagger(\vec{p}) :$ and $: a^\dagger(\vec{k}) a(\vec{p}) :$? Note that a and a^\dagger are the lowering/raising operators of the neutral Klein-Gordon field. [4]
- (b) Consider the scattering of two particles with momenta \vec{p}_1 and \vec{p}_2 into two particles with momenta \vec{p}_3 and \vec{p}_4 . Write down the in- and out-states using the operators a , a^\dagger and the vacuum state $|0\rangle$. [2]

Write down the first two terms of the S -matrix element $\langle \vec{p}_3, \vec{p}_4 | S | \vec{p}_1, \vec{p}_2 \rangle$ and show that the zero-th order term is given by

$$S_0 = (2\pi)^6 4E_{\vec{p}_1} E_{\vec{p}_2} \left[\delta^{(3)}(\vec{p}_4 - \vec{p}_1) \delta^{(3)}(\vec{p}_3 - \vec{p}_2) + \delta^{(3)}(\vec{p}_3 - \vec{p}_1) \delta^{(3)}(\vec{p}_4 - \vec{p}_2) \right] .$$

Give a physical interpretation of the result. Note: you may use the commutation relations for the operators a and a^\dagger given on the FORMULA SHEET. [5]

- (c) Now consider the first order term of the S -matrix element and show that it equals $(-i\lambda)(2\pi)^4 \delta^{(4)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$. Hints: In your derivation you can be sloppy with numerical factors. Furthermore, you may use the following results without proof:

$$: \phi^4(x) : = 6 \int dk dk' dk'' dk''' a^\dagger(\vec{k}) a^\dagger(\vec{k}') a(\vec{k}'') a(\vec{k}''') \exp(i(k + k' - k'' - k''') \cdot x)$$

where dk is shorthand for $d^3k / ((2\pi)^3 2E_{\vec{k}})$, $k \cdot x = E_{\vec{k}} t - \vec{k} \cdot \vec{x}$, and

$$a(\vec{p}_3) a(\vec{p}_4) a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) |0\rangle = S_0 |0\rangle$$

where you can take the expression for S_0 from Question 5(b). [9]

FORMULA SHEET (in units $\hbar = c = 1$)

4-vector notation:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu \eta_{\mu\nu} = a_\mu b_\nu \eta^{\mu\nu} \quad \text{with} \quad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \quad , \quad x_\mu = (t, -\vec{x})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \quad , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right) \quad , \quad \hat{p}^\mu = i\partial^\mu \quad , \quad \hat{p}_\mu = i\partial_\mu$$

Klein-Gordon equation: $(-\hat{p} \cdot \hat{p} + m^2)\psi = (\partial_\mu \partial^\mu + m^2)\psi = (\square + m^2)\psi = 0$

Free Dirac equation in Hamiltonian form: $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha} \cdot \vec{\hat{p}} + \beta m)\Psi$, or in covariant form:

$$(i\hat{\not{p}} - m)\Psi = (i\gamma^\mu \partial_\mu - m)\Psi = (\hat{\not{p}} - m)\Psi = (\gamma \cdot \hat{p} - m)\Psi = (\gamma^\mu \hat{p}_\mu - m)\Psi = 0$$

Dirac and Gamma matrices:

$$\begin{aligned} (\alpha^i)^2 &= \mathbb{I}_4, \quad i = 1, 2, 3; \quad \beta^2 = \mathbb{I}_4; \quad \alpha^i \alpha^j + \alpha^j \alpha^i = 0, \quad i \neq j; \quad \alpha^i \beta + \beta \alpha^i = 0, \quad i \neq j; \\ \gamma^0 &= \beta, \quad \gamma^i = \beta \alpha^i, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}_4, \\ \gamma_5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \end{aligned}$$

Dirac matrices:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3 \quad , \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix},$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that α^i , β and γ^0 are Hermitian, whereas the γ^i are anti-Hermitian. \mathbb{I}_d represents a $d \times d$ identity matrix.

Commutation relations of the raising/lowering operators of the neutral Klein-Gordon field:

$$\left[a(\vec{k}), a(\vec{k}') \right] = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}') \right] = 0, \quad \left[a(\vec{k}), a^\dagger(\vec{k}') \right] = (2\pi)^3 2E_{\vec{k}} \delta^{(3)}(\vec{k} - \vec{k}') \quad \text{with} \quad E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}.$$