

## MSci EXAMINATION

PHY-415 (MSci 4242)      Relativistic Waves and Quantum Fields

Time Allowed: 2 hours 30 minutes

Date: 21st May 2008

Time: 10:00

Instructions:      **Answer THREE QUESTIONS only. Each Question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A FORMULA SHEET is provided at the end of the examination paper.**

**COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.**

Data:              We use units where  $\hbar = c = 1$ . A **FORMULA SHEET** is provided at the end of the examination paper.

Examiners:      **Dr A Brandhuber (CO)**  
                      **Dr S Ramgoolam (DCO)**

**YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR**

**Question 1:** The Dirac equation:

- (a) Give a derivation of the Dirac equation and motivate the form of its ansatz. Derive the continuity equation of the Dirac equation and show that the probability density is given by  $\rho = \Psi^\dagger \Psi$ . What is the main difference between the probability densities of the Klein-Gordon equation and the Dirac equation?

[7]

- (b) Find all plane wave solutions of the Dirac equation for a particle at rest, i.e.  $\vec{p} = 0$ . (You may use the explicit representation of the Dirac matrices given on the formula sheet.) Give a physical interpretation of the solutions. State two alternative methods to generate solutions with arbitrary spatial momentum  $\vec{p}$ .

[4]

- (c) For the Dirac Hamiltonian  $\hat{H} = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m$ , with  $\hat{\vec{p}} = -i\vec{\nabla}$ , show that

$$[\hat{H}, \hat{L}] = -i (\vec{\alpha} \times \hat{\vec{p}}) .$$

Also show that

$$[\hat{H}, \Sigma^i] = \begin{pmatrix} 0 & (\vec{\sigma} \cdot \hat{\vec{p}})\sigma^i - \sigma^i(\vec{\sigma} \cdot \hat{\vec{p}}) \\ (\vec{\sigma} \cdot \hat{\vec{p}})\sigma^i - \sigma^i(\vec{\sigma} \cdot \hat{\vec{p}}) & 0 \end{pmatrix} .$$

Finally, using  $(\vec{\sigma} \cdot \hat{\vec{p}})\sigma^i - \sigma^i(\vec{\sigma} \cdot \hat{\vec{p}}) = 2i (\vec{\sigma} \times \hat{\vec{p}})^i$ , deduce that

$$[\hat{H}, \frac{1}{2}\vec{\Sigma}] = i (\vec{\alpha} \times \hat{\vec{p}}) ,$$

and, hence, that the total angular momentum commutes with the Dirac Hamiltonian.

You may use that  $\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$  where the  $\sigma^i$  are the standard Pauli matrices.

[9]

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**Question 2:** Dirac equation in an electromagnetic field and the magnetic moment of the electron (set  $\hbar = c = 1$ ):

- (a) In classical relativistic mechanics the interaction of a particle carrying charge  $q$  in an external electromagnetic field is obtained by substituting the 4-momentum as  $p^\mu \rightarrow p^\mu + qA^\mu$ , where  $A^\mu$  denotes the electromagnetic 4-vector potential. Hence, find the covariant and the Hamiltonian form of the Dirac equation for a Dirac fermion with charge  $q$  in an external electromagnetic field.

[4]

- (b) Show that the electromagnetic field  $F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$  is invariant under the gauge transformation  $A^\mu \rightarrow A^\mu + \nabla^\mu \Lambda$ , with  $\Lambda$  an arbitrary, real function of the space-time coordinates. How must the Dirac wavefunction  $\Psi$  transform under a gauge transformation, in order that the combined transformation of  $A^\mu$  and  $\Psi$  preserves the form of the Dirac equation in the presence of an electromagnetic field derived in (a), up to an overall phase factor?

[5]

- (c) Consider the non-relativistic limit of the Hamiltonian form of the Dirac equation in the presence of an external electromagnetic field found in (a), using the Dirac matrices as defined in the formula sheet. In this limit we can write the wave function  $\Psi$  in terms of two-component spinors  $\phi$  and  $\chi$  as

$$\Psi = e^{-imt} \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where  $\phi$  and  $\chi$  vary slowly with time. Assuming that  $A^0 = 0$ , show that  $\phi$  obeys the wave equation

$$i \frac{\partial \phi}{\partial t} = \frac{1}{2m} (\vec{\sigma} \cdot \vec{\Pi})^2 \phi,$$

with  $\vec{\Pi} \equiv \hat{p} + q\vec{A} = -i\vec{\nabla} + q\vec{A}$ .

[6]

Using the fact that

$$(\vec{\sigma} \cdot \vec{\Pi})^2 = (\vec{\Pi})^2 + q\vec{\sigma} \cdot \vec{B},$$

with  $\vec{B}$  the magnetic field, derive an expression for the spin magnetic moment of a Dirac particle.

[5]

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**Question 3:** Massless Dirac particles — neutrinos:

In the following use the *chiral representation* of the Dirac matrices

$$\beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad i = 1, 2, 3,$$

where the  $\sigma^i$  denote the Pauli matrices.

- (a) Define the helicity of a particle. What is the form of the helicity operator for a Dirac particle?

[2]

- (b) Describe (in words) how we have to modify the solutions of the Dirac equation to be able to describe massless neutrinos and anti-neutrinos.

[3]

- (c) Consider positive energy, plane wave solutions of the Dirac equation (using the above Dirac matrices)

$$\Psi = e^{-ip \cdot x} \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where  $\phi$  and  $\chi$  denote two component column spinors. Derive equations for  $\phi$  and  $\chi$  for non-zero mass.

[6]

- (d) Derive the equations for  $\phi$  and  $\chi$  in the massless case ( $m = 0$ ). What are the helicities of  $\phi$  and  $\chi$ ?

[4]

- (e) Show how to construct spinors to describe massless neutrinos with helicity  $-\frac{1}{2}$  in terms of a positive energy solution of the massless Dirac equation.

[5]

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**Question 4:** The Klein-Gordon field

- (a) The free, neutral Klein-Gordon field  $\phi = \phi^\dagger$  has Lagrangian density  $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ . Obtain the field equation for  $\phi$  and the Hamiltonian density  $\mathcal{H}$  in terms of  $\phi$  and its derivatives.

[3]

- (b) The free, neutral Klein Gordon field may be expanded in the form

$$\phi = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[ a(\vec{k}) e^{-ik \cdot x} + a^\dagger(\vec{k}) e^{ik \cdot x} \right],$$

with  $E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}$ . Show that the commutation relations

$$\left[ a(\vec{k}), a^\dagger(\vec{k}') \right] = (2\pi)^3 2E_{\vec{k}} \delta^{(3)}(\vec{k} - \vec{k}'), \quad \text{and} \quad \left[ a(\vec{k}), a(\vec{k}') \right] = \left[ a^\dagger(\vec{k}), a^\dagger(\vec{k}') \right] = 0$$

imply the equal time commutation relation

$$[\phi(t, \vec{x}), \Pi(t, \vec{x}')] = i\delta^{(3)}(\vec{x} - \vec{x}'),$$

where  $\Pi = \dot{\phi}$  denotes the momentum canonically conjugate to  $\phi$ .

[6]

- (c) The Hamiltonian for the real Klein-Gordon field can be written in the form

$$H = \frac{1}{2} \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} E_{\vec{k}} \left[ a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}) \right].$$

Show that the vacuum expectation value of the Hamiltonian, i.e. the vacuum energy  $\langle 0|H|0\rangle$  is infinite, where the vacuum state is defined as the state for which  $a(\vec{k})|0\rangle = 0$  for all  $\vec{k}$ . Describe the prescription with which this infinity is removed in quantum field theory.

[4]

- (d) Show that  $\left[ H, a^\dagger(\vec{k}) \right] = E_{\vec{k}} a^\dagger(\vec{k})$  and  $\left[ H, a(\vec{k}) \right] = -E_{\vec{k}} a(\vec{k})$ . Hence, what is the physical interpretation of the operators  $a^\dagger(\vec{k})$  and  $a(\vec{k})$ ?

[4]

- (e) Describe qualitatively what goes wrong if we try to quantise Dirac fermions using equal time commutation relations and state how the quantisation procedure has to be modified to avoid the problems?

[3]

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**Question 5:** The S-matrix

- (a) Assume that the Hamiltonian  $H$  is split up in a free and interacting part as  $H = H_0 + H_{int}$ . The interaction Hamiltonian in the interaction picture is given as

$$H_I \equiv e^{iH_0 t} (H_{int})_S e^{-iH_0 t}$$

Show that a state  $|\psi(t)\rangle_I$  in the interaction picture obeys the Schrödinger equation

$$i \frac{d|\psi(t)\rangle_I}{dt} = H_I(t) |\psi(t)\rangle_I.$$

(Note that states and operators in the Schrödinger picture (subscript  $S$ ) and in the interaction picture (subscript  $I$ ) are related as:  $|\psi(t)\rangle_I = e^{iH_0 t} |\psi(t)\rangle_S$  and  $\mathcal{O}_I(t) = e^{iH_0 t} \mathcal{O}_S e^{-iH_0 t}$ .)

[5]

- (b) Write the solution of the Schrödinger equation in Question 5(a) as

$$|\psi(t)\rangle_I = U(t, t_0) |\psi(t_0)\rangle_I,$$

where  $U(t, t_0)$  is the unitary time evolution operator with  $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$ . Hence, find a differential equation for  $U(t, t_0)$  and from that the integral representation of this equation imposing the condition  $U(t, t) = 1$ . The solution to this equation is given by Dyson's formula

$$U(t, t_0) = T \exp \left( -i \int_{t_0}^t H_I(t') dt' \right)$$

where  $T$  is the time ordering operator. Show that Dyson's formula is a solution to the differential equation you derived for  $U(t, t_0)$ .

[10]

- (c) State how the S-matrix (operator)  $S$  is related to the operator  $U(t, t_0)$ . Give a definition of initial and final states in a scattering process and write S-matrix elements or scattering amplitudes in terms of  $S$ , initial states and final states. Give a qualitative description of how physical cross sections are obtained from a given S-matrix element or scattering amplitude.

[5]

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## Formula Sheet (in units $\hbar = c = 1$ )

4-vector notation:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} = a_\mu b_\nu g^{\mu\nu} \quad \text{with} \quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \quad , \quad x_\mu = (t, -\vec{x})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right) \quad , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \vec{\nabla} \right) \quad , \quad \hat{p}^\mu = i\partial^\mu \quad , \quad \hat{p}_\mu = i\partial_\mu$$

Klein-Gordon equation:  $(-\hat{p} \cdot \hat{p} + m^2)\psi = (\partial_\mu \partial^\mu + m^2)\psi = (\square + m^2)\psi = 0$

Free Dirac equation in Hamiltonian form:  $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha} \cdot \hat{\vec{p}} + \beta m)\Psi$ , or in covariant form:

$$(i\hat{\not{p}} - m)\Psi = (i\gamma^\mu \partial_\mu - m)\Psi = (\hat{\not{p}} - m)\Psi = (\gamma \cdot \hat{p} - m)\Psi = (\gamma^\mu \hat{p}_\mu - m)\Psi = 0$$

Dirac and Gamma matrices:

$$\begin{aligned} (\alpha^i)^2 &= \mathbb{I}, \quad i = 1, 2, 3; \quad \beta^2 = \mathbb{I}; \quad \alpha^i \alpha^j + \alpha^j \alpha^i = 0, \quad i \neq j; \quad \alpha^i \beta + \beta \alpha^i = 0, \quad i \neq j; \\ \gamma^0 &= \beta, \quad \gamma^i = \beta \alpha^i, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{I}, \\ \gamma_5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \end{aligned}$$

Dirac representation:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3 \quad , \quad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that  $\alpha^i$ ,  $\beta$  and  $\gamma^0$  are Hermitian, whereas the  $\gamma^i$  are anti-Hermitian.

**End of Examination Paper**