MSci 4242 Relativistic Waves & Quantum Fields

Problem Set 6

1. The Lagrangian density for a massive vector field B_μ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 B_\mu B^\mu \,,$$

where the fieldstrength is defined as $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and m is the mass of the vector field. Find the Euler-Lagrange equations of motion for B_ν and show (for $m \neq 0$) that they imply

$$\partial_\nu B^\nu = 0 \,.$$

Is the Lagrangian density invariant under gauge transformations: $B_\nu \to B_\nu + \partial_\nu \Lambda(x)$, where $\Lambda(x)$ is an arbitrary function depending on x^ν?

2. Consider a real scalar field $\phi = \phi^\dagger$ with Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$ and classical Hamiltonian

$$H = \frac{1}{2} \int d^3x \left(\dot{\phi}^2 + |\vec{\nabla}\phi|^2 + m^2 \phi^2 \right) \,.$$

Using the mode expansion

$$\phi = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x} \right] \,,$$

show that the quantum Hamiltonian can be written as

$$H = \frac{1}{2} \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} E_{\vec{k}} \left[a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right] \,.$$

Remember that the $a(\vec{k})$ and $a^\dagger(\vec{k})$ are operators, so their order in a product matters! (Hints: the proof involves use of the identity $E_{\vec{k}}^2 = \vec{k}^2 + m^2$. First work out expressions for $\dot{\phi}$ and $\vec{\nabla}\phi$ and then insert in the classical formula for H. Terms quadratic in $a(\vec{k})$ and quadratic in $a^\dagger(\vec{k})$ cancel separately.)