MSci 4242 Relativistic Waves & Quantum Fields

Problem Set of Week 05

In the following set $\hbar = c = 1$.

1. Under parity $\mathcal{P}$ the wave function transforms as $\Psi \rightarrow \Psi' = P\Psi$ with $P = \gamma^0$. Show that $PU(\vec{p}, s) = U(-\vec{p}, s)$ and $PV(\vec{p}, s) = -V(-\vec{p}, s)$. Also show for $C = i\gamma^2\gamma^0$ that $C\gamma^0 U^*(p, 1) = V(p, 1)$.

2. The covariant form of the Dirac equation in the presence of an electromagnetic field is
$$(i\nabla\!\!\!/ + qA\!\!\!/ - m)\Psi = 0\,.$$
Show that the charge conjugated wave function $\Psi_C = C\gamma^0\Psi^*$, with $C = i\gamma^2\gamma^0$, obeys the equation
$$(i\nabla\!\!\!/ - qA\!\!\!/ - m)\Psi_C = 0\,,$$
i.e. the charge conjugated wave function describes a particle with opposite charge.

3. The non-relativistic limit of the Dirac equation is given by the Pauli equation
$$i\frac{\partial \phi}{\partial t} = \left(\frac{(\widehat{\vec{p}} + q\vec{A})^2}{2m} + \frac{q}{2m}\vec{\sigma} \cdot \vec{B} - qA^0 \right) \phi\,.$$
Turn on a weak and homogeneous magnetic field with vector potential $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$ where $\vec{B}$ is constant. Neglecting terms quadratic in $\vec{A}$ show that the Pauli equation can be written in the form
$$i\frac{\partial \phi}{\partial t} = \left(\frac{(\widehat{\vec{p}})^2}{2m} + \frac{q}{2m}(\widehat{\vec{L}} + 2\widehat{\vec{S}}) \cdot \vec{B} - qA^0 \right) \phi\,.$$