MSci 4242 Relativistic Waves & Quantum Fields

Problem Set of Week 04

In the following set $\hbar = c = 1$.

1. Show explicitly, using the positive energy plane wave solutions $U(p,s)$ of the Dirac equation constructed in the lecture, that $U^\dagger(p,s)U(p,s) = 2E$ and $\overline{U}(p,s)U(p,s) = 2m$. For this you will need to prove that $(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^2\mathbb{I}_2$.

2. Show that the free Dirac equation is invariant under time reversal $\mathcal{T}$: $t \to t' = -t$, $\vec{x} \to \vec{x}' = \vec{x}$ and $\Psi \to \Psi'(x') = T\Psi^*(x)$ with $T = -\gamma^1\gamma^3$. Hints: γ^0, γ^1 and γ^3 are real matrices; γ^2 is imaginary; to simplify the strings of gamma-matrices at intermediate stages use the anti-commutation relations and the fact that $(\gamma^0)^2 = \mathbb{I}$ and $(\gamma^i)^2 = -\mathbb{I}$.

3. Find the transformation behaviour of $\overline{\Psi}\gamma_5\Psi$ under space inversion $\mathcal{P}$ and of $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ under charge conjugation $\mathcal{C}$.

4. Show that the plane wave solution of the Dirac equation $U(\vec{p}=0,s)$ is related to the plane wave solution $U(\vec{p}=(0,0,p_z),s)$ by a boost in the z-direction. The boost of a Dirac spinor in the z-direction is given by the matrix

$$S = e^{-i\omega\sigma(K_z)} \quad \text{, where } \sigma(K_z) = \frac{i}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} .$$

   Show that the correct boost parameter ω obeys $\cosh\frac{\omega}{2} = \sqrt{\frac{E+m}{2m}}$ and $\sinh\frac{\omega}{2} = \frac{p_z}{\sqrt{2m(E+m)}}$. Remember, that E, p_z and m are related as $E^2 = p_z^2 + m^2$ (since $p_x = p_y = 0$ in this example).