MSci 4242 Relativistic Waves & Fields

Problem Set of Week 03

1. The gamma-matrices γ^μ are defined in terms of the Dirac matrices as $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i, i = 1, 2, 3$. Use the properties of β and α^i to show that
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_4\,,$$
where $\mathbb{I}_4$ denotes the four-by-four unit matrix.

2. Some Gamma-matrix gymnastics:
Show that: $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$
Calculate: $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
Show that: $\gamma^0\gamma_5\gamma^0 = -\gamma_5 = -\gamma_5^\dagger$ and $\gamma^0(\gamma_5\gamma^\mu)\gamma^0 = (\gamma_5\gamma^\mu)^\dagger$

3. Spin and Dirac Equation: For the Dirac Hamiltonian $\widehat{H} = c\vec{\alpha}\cdot\widehat{\vec{p}}+\beta mc^2$, with $\widehat{\vec{p}} = -i\hbar\vec{\nabla}$, show that
$$[\widehat{H}, \widehat{\vec{L}}] = -i\hbar c\left(\vec{\alpha}\times\widehat{\vec{p}}\right)\,. \qquad (1)$$
Also show that
$$[\widehat{H}, \Sigma^i] = c\begin{pmatrix} 0 & (\vec{\sigma}\cdot\widehat{\vec{p}})\sigma^i - \sigma^i(\vec{\sigma}\cdot\widehat{\vec{p}}) \\ (\vec{\sigma}\cdot\widehat{\vec{p}})\sigma^i - \sigma^i(\vec{\sigma}\cdot\widehat{\vec{p}}) & 0 \end{pmatrix}\,, \qquad (2)$$
and that $(\vec{\sigma}\cdot\widehat{\vec{p}})\sigma^i - \sigma^i(\vec{\sigma}\cdot\widehat{\vec{p}}) = 2i\left(\vec{\sigma}\times\widehat{\vec{p}}\right)^i$. Finally, deduce that
$$[\widehat{H}, \frac{\hbar}{2}\vec{\Sigma}] = i\hbar c\left(\vec{\alpha}\times\widehat{\vec{p}}\right)\,, \qquad (3)$$
and, hence, that the total angular momentum commutes with the Dirac Hamiltonian.