MSci 4242/QMUL PHY-415 Relativistic Waves & Quantum Fields

Problem Set of Week 02

1. A finite boost in the x-direction is given by the matrix

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

In the lecture we found for an infinitesimal boost parameter $\omega \ll 1$ the approximate expression $\Lambda = \mathbb{I}_4 - i\omega K_x$, where $\mathbb{I}_4$ denotes a 4×4 unit matrix, and

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad .$$

denotes the generator of boosts in the x-direction. Show that the finite Lorentz transformation Λ can be obtained by *exponentiating* the infinitesimal Lorentz transformation, i.e. show that

$$\Lambda = e^{-i\omega K_x} \quad .$$

Note: The exponential function is defined by its usual power series expansion

$$e^{-i\omega K_x} = \mathbb{I}_4 + (-i\omega K_x) + \frac{1}{2!}(-i\omega K_x)^2 + \frac{1}{3!}(-i\omega K_x)^3 + \ldots \quad ,$$

where the n-th power of a matrix is defined through matrix multiplication.

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2. Using the explicit form of the generators for boost

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

and rotations

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

check that the following commutation relations are obeyed:

$$\begin{aligned} [J_x, J_y] &= iJ_z \\ [K_x, K_y] &= -iJ_z \\ [J_x, K_y] &= iK_z \end{aligned}$$

3. Under a Lorentz transformation the electromagnetic field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

transforms as $F^{\mu\nu} \rightarrow F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$. Calculate explicitly the Lorentz transformation of $F^{\mu\nu}$ under a boost in the x-direction given by $\Lambda^\mu{}_\nu$ as defined in Exercise 1. Note that $E^i, i = 1,2,3$ and $B^i, i = 1,2,3$ correspond to the vector components of the electric field $\vec{E}$ and magnetic field $\vec{B}$. Hint : Write the Lorentz transformation first in the form of matrix multiplications.

Hence, find expressions for the Lorentz-transformed electric and magnetic fields, $\vec{E'}$ and $\vec{B}'$, in terms of the components of $\vec{E}$ and $\vec{B}$ and $\gamma = \cosh\omega$ and $\beta = \tanh\omega$.