MSci 4242/QMUL PHY-415 Relativistic Waves and Quantum Fields

Problem Set of Week 01

(For convenience we drop the ^ for QM operators in the following.)

1. Using Cartesian coordinates show that the three components L_x, L_y and L_z of the AM operator $\vec{L}$ obey the standard $SO(3)$ algebra:

   $$[L_x, L_y] = i\hbar L_z \tag{1}$$

   plus cyclic permutations of x, y, z.

   Furthermore show that the three components of the AM operator commute with $\vec{\nabla}^2$ and $r^2 = x^2 + y^2 + z^2$. What is the consequence of this result?

2. Consider a system with orbital AM operator $\vec{L}$, spin operator $\vec{S}$ and total AM operators $\vec{J} = \vec{L} + \vec{S}$. The components of $\vec{L}$ and $\vec{S}$ obey the same commutator algebra as in eqn. (1), and, furthermore, commute with each other i.e. $[L_x, S_x] = [L_x, S_y] = \ldots = 0$. In the Lecture we came across the Spin-Orbit interaction, which is a relativistic effect and is proportional to the operator (up to an r dependent factor)

   $$A = \vec{L} \cdot \vec{S}\,. \tag{2}$$

   Compute the commutators

   $$[\vec{L}, A] = ? \tag{3}$$

   $$[\vec{S}, A] = ? \tag{4}$$

   and show that A and $\vec{J}$ commute i.e.

   $$[\vec{J}, A] = 0. \tag{5}$$

   Discuss the significance of the last equation.

3. Which components of the momentum operator $\vec{P}$ and the AM operator $\vec{L}$ correspond to conserved quantities (constants of motion) for the following QM Hamiltonian?

   $$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + Cz^2 \tag{6}$$

   where C is a positive real constant.