MSci EXAMINATION

PHY-414 (MSci 4241) Relativistic Quantum Mechanics

Time Allowed: 2 hours 30 minutes

Date:

Time:

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

Data: We use units where $\hbar = c = 1$. A formula sheet is provided at the end of the paper.

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

 Version 1.1

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Question 1: The Dirac equation:

(a) Give a derivation of the Dirac equation and motivate the form of its ansatz.

[5]

(b) Derive the continuity equation of the Dirac equation and show that the probability density is given by $\rho = \Psi^\dagger \Psi$. What is the main difference between the probability densities of the Klein-Gordon equation and the Dirac equation?

[3]

(c) Find all plane wave solutions of the Dirac equation for a particle at rest, i.e. $\vec{p} = 0$. Give a physical interpretation of the solutions. State two alternative methods to generate solutions with arbitrary spatial momentum $\vec{p}$.

[4]

(d) Write down the covariant form of the Dirac equation. Assume that Ψ transforms under a Lorentz transformation as $\Psi(x) \to \Psi'(x') = S(\Lambda)\Psi(x)$, with $x' = \Lambda x$ and $S(\Lambda)$ a four-by-four matrix. Show that the Dirac equation is form invariant (and hence covariant) if

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu \gamma^\mu \,.$$

[6]

(e) Consider adding a term of the form $T_{\mu_1\mu_2}\gamma^{\mu_1}\gamma^{\mu_2}\Psi$ to the covariant form of the Dirac equation, where the tensor $T_{\mu_1\mu_2}$ transforms covariantly under Lorentz transformations. Does such a term spoil the covariance of the Dirac equation?

[2]

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Question 2: Dirac equation in an electromagnetic field and the magnetic moment of the electron (set $\hbar = c = 1$):

(a) In classical relativistic mechanics the interaction of a particle carrying charge q in an external electromagnetic field is obtained by substituting the 4-momentum as $p^\mu \to p^\mu + qA^\mu$, where A^μ denotes the electromagnetic 4-vector potential. Hence, find the covariant and the Hamiltonian form of the Dirac equation for a Dirac fermion with charge q in an external electromagnetic field.

[4]

(b) Show that the electromagnetic field $F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$ is invariant under the gauge transformation $A^\mu \to A^\mu + \nabla^\mu \Lambda$, with Λ an arbitrary, real function of the space-time coordinates. How must the Dirac wavefunction Ψ transform under a gauge transformation, in order that the combined transformation of A^μ and Ψ preserves the form of the Dirac equation in the presence of an electromagnetic field derived in (a), up to an overall phase factor?

[5]

(c) Consider the non-relativistic limit of the Hamiltonian form of the Dirac equation in the presence of an external electromagnetic field found in (a), using the Dirac matrices as defined in the formula sheet. In this limit we can write the wave function Ψ in terms of two-component spinors ϕ and χ as

$$\Psi = e^{-imt} \begin{pmatrix} \phi \\ \chi \end{pmatrix} ,$$

where ϕ and χ vary slowly with time. Assuming that $A^0 = 0$, show that ϕ obeys the wave equation

$$i\frac{\partial \phi}{\partial t} = \frac{1}{2m} \left(\vec{\sigma} \cdot \vec{\Pi}\right)^2 \phi ,$$

with $\vec{\Pi} \equiv \hat{\vec{p}} + q\vec{A} = -i\vec{\nabla} + q\vec{A}$.

[6]

Using the fact that

$$\left(\vec{\sigma} \cdot \vec{\Pi}\right)^2 = \left(\vec{\Pi}\right)^2 + q\vec{\sigma} \cdot \vec{B} ,$$

with $\vec{B}$ the magnetic field, derive an expression for the spin magnetic moment of a Dirac particle.

[5]

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Question 3: Massless Dirac particles — neutrinos:
In the following use the *chiral representation* of the Dirac matrices

$$\beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \,, \; \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \; i = 1, 2, 3 \,,$$

where the σ^i denote the Pauli matrices.

(a) Define the helicity of a particle. What is the form of the helicity operator for a Dirac particle?

[2]

(b) Describe (in words) how we have to modify the solutions of the Dirac equation to be able to describe massless neutrinos and anti-neutrinos.

[3]

(c) Consider positive energy, plane wave solutions of the Dirac equation (using the above Dirac matrices)

$$\Psi = e^{-ip\cdot x} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \,,$$

where ϕ and χ denote two component column spinors. Derive equations for ϕ and χ for non-zero mass.

[6]

(d) Derive the equations for ϕ and χ in the massless case ($m = 0$). What are the Helicities of ϕ and χ?

[4]

(e) Show how to construct spinors to describe massless neutrinos with helicity $-\frac{1}{2}$ in terms of a positive energy solution of the massless Dirac equation.

[5]

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Question 4: The Dirac propagator:

(a) Show that for $p^2 \neq m^2$ the momentum space propagator for a free relativistic electron is given by

$$\widetilde{S}_F(p) = (\not{p} - m)^{-1}\,.$$

[6]

(b) Show that for $p^2 \neq m^2$ this can be written as

$$\widetilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2}\,.$$

[2]

(c) In order to regularize the singularity at $p^2 = m^2$ we introduced the Feynman prescription for the Dirac propagator

$$\widetilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}\,,$$

where ϵ is a small, positive, real constant. Show that for $t' > t$ the free electron propagator $S_F(x', x)$ contains only positive frequency modes. Briefly discuss the Feynman boundary conditions that lead to this $i\epsilon$ (Feynman) prescription.

[8]

(d) Alternatively, the singularity at $p^2 = m^2$ of the Dirac propagator can be removed with the prescription

$$\widetilde{S}_R(p) = \frac{(\not{p} + m)}{(p^0 + i\epsilon)^2 - \vec{p}^2 - m^2}\,,$$

which is the prescription for the retarded propagator. Analyse carefully the locaction of the poles and the corresponding deformation of the integration contour in the complex p^0 plane. Hence, show that for $t' < t$ the retarded propagator $S_R(x', x)$ vanishes.

[4]

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Question 5: The Klein-Gordon field:

In the following you may assume that $[a(k), a(k')] = \left[a^\dagger(k), a^\dagger(k')\right] = 0$, and $\left[a(k), a^\dagger(k')\right] = \delta^{(3)}(\vec{k} - \vec{k}')$.

(a) The free, neutral Klein-Gordon field $\phi = \phi^\dagger$ has Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$. Obtain the Hamiltonian density $\mathcal{H}$ in terms of ϕ and its derivatives.

[3]

(b) The field may be expanded in the form

$$\phi = \int d^3k \left[a(k) f_k^{(+)}(x) + a^\dagger(k) f_k^{(-)}(x)\right] ,$$

where $f_k^{(\pm)}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} e^{\mp ik\cdot x}$, with $E_k = +\sqrt{\vec{k}^2 + m^2}$. Hence, show that the Hamiltonian can be written in the form

$$H = \frac{1}{2} \int d^3k E_k \left[a(k)a^\dagger(k) + a^\dagger(k)a(k)\right] .$$

[9]

(c) Show that the vacuum expectation value of the Hamiltonian, i.e. the vacuum energy $\langle 0|H|0\rangle$ is infinite, where the vacuum state is defined as the state for which $a(k)|0\rangle = 0$ for all k. Describe the prescription with which this infinity is removed in quantum field theory.

[4]

(d) Also show that $\left[H, a^\dagger(k)\right] = E_k a^\dagger(k)$ and $[H, a(k)] = -E_k a(k)$. What is the physical interpretation of the operators $a^\dagger(k)$ and $a(k)$?

[4]

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Formula Sheet (in units $\hbar = c = 1$)

4-vector notation:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} = a_\mu b_\nu g^{\mu\nu} \text{ with } g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \ , \ x_\mu = (t, -\vec{x})$$

$$\nabla^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \ , \ \nabla_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \ , \ \widehat{p}^\mu = i\nabla^\mu \ , \ \widehat{p}_\mu = i\nabla_\mu$$

Klein-Gordon equation: $(-\widehat{p} \cdot \widehat{p} + m^2)\psi = (\nabla_\mu \nabla^\mu + m^2)\psi = (\Box + m^2)\psi = 0$

Free Dirac equation in Hamiltonian form: $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m)\Psi$, or in covariant form:

$$(\widehat{\not{p}} - m)\Psi = (\gamma \cdot \widehat{p} - m)\Psi = (\gamma^\mu \widehat{p}_\mu - m)\Psi = (i\not{\nabla} - m)\Psi = (i\gamma^\mu \nabla_\mu - m)\Psi = 0$$

Dirac and Gamma matrices:

$$\begin{aligned} &(\alpha^i)^2 = \mathbb{I}\,,\ i = 1,2,3;\ \beta^2 = \mathbb{I};\ \alpha^i\alpha^j + \alpha^j\alpha^i = 0\,,\ i \neq j;\ \alpha^i\beta + \beta\alpha^i = 0\,,\ i \neq j; \\ &\gamma^0 = \beta,\ \gamma^i = \beta\alpha^i,\ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}\,, \\ &\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \end{aligned} \tag{1}$$

Dirac representation:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},\ i = 1,2,3 \ , \ \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that α^i, β and γ^0 are Hermitian, whereas the γ^i are anti-Hermitian.

End of Examination Paper **Dr A Brandhuber**