MSci EXAMINATION

PHY-414 (MSci 4241) Relativistic Quantum Mechanics

Time Allowed: 2 hours 30 minutes

Date:

Time:

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

Data: We use units where $\hbar = c = 1$. A formula sheet is provided at the end of the paper.

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

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Question 1: Angular momenta $\widehat{\vec{J}}_1$ and $\widehat{\vec{J}}_2$ are combined to total angular momentum (set $\hbar = 1$)

$$\widehat{\vec{J}} = \widehat{\vec{J}}_1 + \widehat{\vec{J}}_2 \,.$$

(a) Derive the maximum value of the quantum number j for $\widehat{\vec{J}}^2$ in terms of the quantum numbers j_1 and j_2 for $(\widehat{\vec{J}}_1)^2$ and $(\widehat{\vec{J}}_2)^2$.

[4]

(b) Show that $[\widehat{J}_-, \widehat{J}_z] = \widehat{J}_-$ using the standard commutator algebra for angular momentum operators and $\widehat{J}_- = \widehat{J}_x - i\widehat{J}_y$. Use this result to show that $\widehat{J}_-|j,m\rangle$ is proportional to the eigenstate $|j, m-1\rangle$.

[3]

(c) Angular momenta $j_1 = k$, where k is a positive integer or half-integer, and $j_2 = 1$ are combined. Construct the following eigenstates of $\widehat{\vec{J}}^2$ and $\widehat{J}_z$:

$|k+1, k+1\rangle$ **[3]**

$|k+1, k\rangle$ **[4]**

$|k, k\rangle$ **[6]**

You may assume that $\widehat{J}_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}|j,m-1\rangle$.

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Question 2: The Dirac equation and charge conjugation

(a) Describe what is meant by a symmetry of a wave equation.

[2]

(b) Show that the free Dirac equation is invariant under charge conjugation $\mathcal{C}$, where $\mathcal{C}$ acts trivially on space time coordinates and on the wavefunction as $\Psi \rightarrow \Psi_C = C\gamma^0\Psi^*$ with $C = i\gamma^2\gamma^0$.

[8]

(c) Determine the behaviour under charge conjugation of the Dirac covariants $\overline{\Psi}\gamma^\mu\Psi$ and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$.

[8]

(d) Hence, discuss why charge conjugation invariance is broken by the weak interactions.

[2]

(You may *assume* that $C^\dagger = -C$, $C^2 = -\mathbb{I}$, $C\gamma^0(\gamma^\mu)^* = -\gamma^\mu(C\gamma^0)$, $\gamma^0 C\gamma^0 = -C$ and $\gamma^\mu C = -C\,(\gamma^\mu)^T$ where T denotes transpose and $\dagger$ denotes Hermitian conjugation. You may also assume that $\gamma_5^\dagger = \gamma_5^T = \gamma_5$ and $\{\gamma^\mu, \gamma_5\} = 0$.)

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Question 3: Massless Dirac particles — neutrinos:
In the following use the *chiral representation* of the Dirac matrices

$$\beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \ , \ \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \ i = 1, 2, 3 \, ,$$

where the σ^i denote the Pauli matrices.

(a) Define the helicity of a particle. What is the form of the helicity operator for a Dirac particle?

[2]

(b) Describe (in words) how we have to modify the solutions of the Dirac equation to be able to describe massless neutrinos and anti-neutrinos.

[3]

(c) Consider positive energy, plane wave solutions of the Dirac equation (using the above Dirac matrices)

$$\Psi = e^{-ip\cdot x} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \, ,$$

where ϕ and χ denote two component column spinors. Derive equations for ϕ and χ for non-zero mass.

[6]

(d) Derive the equations for ϕ and χ in the massless case ($m = 0$). What are the Helicities of ϕ and χ?

[4]

(e) Show how to construct spinors to describe massless neutrinos with helicity $-\frac{1}{2}$ in terms of a positive energy solution of the massless Dirac equation.

[5]

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Question 4: The Dirac propagator:

(a) Show that for $p^2 \neq m^2$ the momentum space propagator for a free relativistic electron is given by

$$\widetilde{S}_F(p) = (\not{p} - m)^{-1} .$$

[6]

(b) Show that for $p^2 \neq m^2$ this can be written as

$$\widetilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2} .$$

[2]

(c) In order to regularize the singularity at $p^2 = m^2$ we introduced the Feynman prescription for the Dirac propagator

$$\widetilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon} ,$$

where ϵ is a small, positive, real constant. Show that for $t' < t$ the free electron propagator $S_F(x', x)$ contains only negative frequency modes. Briefly discuss the Feynman boundary conditions that lead to this $i\epsilon$ (Feynman) prescription.

[6]

(d) The propagator $\widehat{S}_F(x', x)$ for an electron in an electro-magnetic 4-potential A_μ satisfies

$$(i\not{\nabla}' - e\not{A}' - m\mathbb{I})\widehat{S}_F(x', x) = \delta^4(x' - x)\mathbb{I} .$$

Derive an integral equation for $\widehat{S}_F(x', x)$ and, hence, show to first order in e that

$$\widehat{S}_F(x', x) = S_F(x', x) + e \int d^4x_1 S_F(x', x_1)\not{A}(x_1)S_F(x_1, x) ,$$

where $S_F(x', x)$ denotes the free electron propagator.

[6]

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Question 5: Scattering and Feynman rules:

(a) Describe briefly the physical meaning of the scattering amplitude S_{fi}. For a relativistic electron scattering in an electro-magnetic field, write down (without proof) the expression for S_{fi} to first order in the interaction in propagator theory and explain the quantities appearing in the expression.

[4]

(b) Write down the relation between the scattering amplitude S_{fi} and the invariant amplitude M_{fi} (also called invariant matrix element). Only state the result, no proof is needed. For concreteness you may use the example of a scattering process of two Dirac particles into two Dirac particles.

[3]

(c) State the Feynman rules (in momentum space) for processes involving electrons, positrons and photons to calculate M_{fi}. Draw the tree-level Feynman diagrams for electron-electron scattering into two electrons.

[7]

(d) For the case of electron-positron scattering into electron-positron, draw all contributing tree-level Feynman diagrams and determine M_{fi} or S_{fi} using the Feynman rules. (Alternatively, you may use propagator theory to find S_{fi}.)

[6]

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Formula Sheet (in units $\hbar = c = 1$)

Four-vectors:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} = a_\mu b_\nu g^{\mu\nu} \text{ with } g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \;\;,\;\; x_\mu = (t, -\vec{x})$$

$$\nabla^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \;\;,\;\; \nabla_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \;\;,\;\; \widehat{p}^\mu = i\nabla^\mu \;\;,\;\; \widehat{p}_\mu = i\nabla_\mu$$

Klein-Gordon equation: $(-\widehat{p} \cdot \widehat{p} + m^2)\psi = (\nabla_\mu \nabla^\mu + m^2)\psi = (\Box + m^2)\psi = 0$

Free Dirac equation in Hamiltonian form: $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m)\Psi$, or in covariant form:

$$(\widehat{\not{p}} - m)\Psi = (\gamma \cdot \widehat{p} - m)\Psi = (\gamma^\mu \widehat{p}_\mu - m)\Psi = 0$$

Dirac and Gamma matrices:

$$\begin{aligned} &(\alpha^i)^2 = \mathbb{I}\,,\; i = 1,2,3;\;\; \beta^2 = \mathbb{I};\;\; \alpha^i\alpha^j + \alpha^j\alpha^i = 0\,,\; i \neq j;\;\; \alpha^i\beta + \beta\alpha^i = 0\,,\; i \neq j; \\ &\gamma^0 = \beta,\;\; \gamma^i = \beta\alpha^i,\;\; \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}\,, \\ &\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \end{aligned} \tag{1}$$

Dirac representation:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},\; i = 1,2,3 \;\;,\;\; \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\;,\;\; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;\;,\;\; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that α^i, β and γ^0 are Hermitian, wheras the γ^i are anti-Hermitian.

End of Examination Paper **Dr A Brandhuber**