Queen Mary
University of London

MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date:

Time:

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

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1. The electric dipole contribution to the vector potential is given by

$$\mathbf{A} = -\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{ik}{c}\mathbf{p}.$$

(a) Define k and r in this equation. [2 marks]

(b) What is the magnetic field $\mathbf{B}$? [4 marks]

(c) Show that in the far zone $kr >> 1$, with unit vector $\mathbf{n} = \mathbf{r}/r$,

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{k^2}{c}\mathbf{n}\times\mathbf{p},$$
$$\mathbf{E} = c\,\mathbf{B}\times\mathbf{n}$$

[8 marks]

(d) From the expression

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0 c}|r\mathbf{E}|^2$$

for the power radiated per unit solid angle, show that the total power radiated in all directions by pure electric dipole radiation is equal to

$$\frac{\mu_0}{12\pi c}|\mathbf{p}|^2\omega^4$$ [6 marks]

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2. The symmetric stress-energy-momentum tensor $\Theta^{\alpha\beta}$ is defined by

$$\Theta^{\alpha\beta} = \frac{1}{\mu_0}[F^{\alpha\lambda}F_\lambda{}^\beta - \frac{1}{4}\eta^{\alpha\beta}F^{\mu\lambda}F_{\lambda\mu}].$$

(a) Show that

$$\Theta^{00} = \frac{1}{2}\epsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2),$$

and

$$\Theta^{0i} = \frac{1}{c\mu_0}(\mathbf{E}\times\mathbf{B})^i,$$

and give the physical significance of these quantities. [8 marks]

(b) Prove

$$\partial_\alpha\Theta^{\alpha\beta} = j_\alpha F^{\alpha\beta}.$$

[6 marks]

(c) For $\beta = 0$ show that this becomes Poynting's equation

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{j}.$$

[2 marks]

(d) What is the physical meaning of this equation? [4 marks]

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3. (a) Consider spherical coordinates (r, θ, ϕ), and a vector field $\mathbf{A}$ with components

$$A_r = 0, \quad A_\theta = 0, \quad A_\phi = \frac{g(1 - \cos\theta)}{4\pi r \sin\theta}.$$

Show that this potential gives a magnetic field

$$\mathbf{B} = \frac{g}{4\pi r^2}\,\hat{\mathbf{r}}.$$

[5 marks]

(b) Explain what this magnetic field describes. [3 marks]

(c) The Lorentz force law for the motion in this field of a particle of electric charge e, rest mass m, velocity $\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ and momentum $\mathbf{p} = \gamma(v)m\mathbf{v}$ gives

$$\dot{\mathbf{p}} = \frac{eg}{4\pi}\frac{\mathbf{v} \times \mathbf{r}}{r^3}.$$

Show that the quantities

$$E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}, \qquad \mathbf{J} = \gamma(v) m \mathbf{r} \times \mathbf{v} - \frac{eg}{4\pi}\frac{\mathbf{r}}{r},$$

are constants of the motion and explain what these invariants are physically and what the separate terms in $\mathbf{J}$ represent. [8 marks]

(d) Consider the case where the particle in part (c) above is stationary. Assuming that $\mathbf{J}$ has the properties of intrinsic angular momentum, derive the quantisation condition

$$\frac{eg}{4\pi} = \frac{n}{2}\hbar, \qquad n = 0, \pm 1, \pm 2, ...$$

[4 marks]

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4. The Liénard-Wiechert potentials for the electromagnetic fields generated by a charge q following a trajectory $\mathbf{r} = \mathbf{r}(t)$, with instantaneous velocity $\mathbf{u} = \frac{d\mathbf{r}}{dt} = c\boldsymbol{\beta}$, are

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}},$$
$$\mathbf{A} = \frac{\mu_0 q c}{4\pi}\left[\frac{\boldsymbol{\beta}}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}}.$$

(a) Explain the meaning of the notation $[\ldots]_{\text{ret}}$, and define the distance R and the direction vector $\mathbf{n}$. [4 marks]

(b) If $|\boldsymbol{\beta}| << 1$, show that at large distances from the charge the electric field is

$$\mathbf{E}_{\mathbf{far}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{1}{R}(\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}))\right]_{\text{ret}}.$$ [6 marks]

(c) Assuming that the corresponding magnetic field is given by

$$\mathbf{B}_{\mathbf{far}} = \left[\mathbf{n}\times\mathbf{E}_{\mathbf{far}}\right]_{\mathbf{far}}/c,$$

show that at large distances, the Poynting energy-flux vector is

$$\mathbf{S}_{\mathbf{far}} = \frac{1}{\mu_o c}|\mathbf{E}_{\mathbf{far}}|^2\mathbf{n}.$$ [4 marks]

(d) Derive the Larmor formula

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}|\dot{\mathbf{u}}|^2$$

for the total instantaneous power radiated by a non-relativistic accelerated charge. [6 marks]

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5. (a) Show that Maxwell's equations imply that the current is conserved, $\partial_\beta j^\beta = 0$.
[4 marks]

(b) Consider the gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha \Lambda$, for some function $\Lambda(x)$. Explain how the Lagrangian

$$L = \int d^4x \Big(-\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - A_\alpha j^\alpha \Big)$$

is invariant under this transformation. [5 marks]

(c) Write down the equation for the Lorentz force on a particle of charge q moving with velocity $\mathbf{v}$. Explain how this may be generalised to the equation

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B}$$

for the force per unit volume on a charge and current density. [5 marks]

(d) Show that

$$f^k = F^{k\alpha} j_\alpha, \qquad \text{for } k = 1, 2, 3,$$

and define f^0 such that f^α is a four vector. [6 marks]

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Formula Sheet

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c},\\
\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a},\\
\nabla\times(\psi\mathbf{a}) &= (\nabla\psi)\times\mathbf{a}+\psi(\nabla\times\mathbf{a}),\\
\nabla\times(\nabla\times\mathbf{a}) &= \nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a},\\
\nabla\big(\psi(r)\big) &= \mathbf{n}\psi'(r).
\end{aligned}$$

Maxwell's equations:

$$\nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t};$$
$$\nabla\cdot\mathbf{D}=\rho,\qquad \nabla\times\mathbf{H}=\mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}.$$

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

For linear isotropic media:

$$\mathbf{D}=\epsilon\mathbf{E}=\epsilon_0\mathbf{E}+\mathbf{P},\qquad \mathbf{H}=\frac{1}{\mu}\mathbf{B}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}.$$

$$c^2d\tau^2=c^2dt^2-dx^2-dy^2-dz^2=dx^\alpha\eta_{\alpha\beta}dx^\beta.$$

$$\eta_{\alpha\beta}=\begin{cases}+1 & \text{if } \alpha=\beta=0\\ -1 & \text{if } \alpha=\beta=1,2,3\\ 0 & \text{if } \alpha\neq\beta\end{cases}$$

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\nabla\right),\qquad \partial^\mu=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right).$$

$$\partial_\alpha F^{\alpha\beta}=\partial_\alpha\partial^\alpha A^\beta-\partial^\beta\partial_\alpha A^\alpha=\mu_0 j^\beta;\qquad F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0.$$

$$\|F^{\alpha\beta}\|=\begin{pmatrix}0 & -E^1/c & -E^2/c & -E^3/c\\ E^1/c & 0 & -B^3 & B^2\\ E^2/c & B^3 & 0 & -B^1\\ E^3/c & -B^2 & B^1 & 0\end{pmatrix}.$$

In spherical coordinates (r,θ,ϕ), for a vector field $\mathbf{A}$ with components (A_r,A_θ,A_ϕ),

$$\begin{aligned}\nabla\times\mathbf{A}= &\ \hat{\mathbf{r}}\,\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)+\hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\\ &+\hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\end{aligned}$$

End of Examination Paper **Dr WJ Spence**