Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-325 QUANTUM MECHANICS AND SYMMETRY

Time Allowed: 2 hours 15 minutes

Date: 15 May, 2007

Time: 10.00

Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 40 % of the marks, each question in section B carries 30 %. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

Useful Formulae

Integral representation of Dirac Delta function.

$$2\pi\delta(k') = \int dx' e^{ik'x'}$$

Identity operator in terms of continuous eigenstates:

$$1 = \int dx' |x'\rangle\langle x'|$$

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$e^{i\theta\sigma_3} = \cos\theta + i\sigma_3 \sin\theta$

Eigenvalue of the Casimir of a spin$j$ representation is $j(j+1)$.

Dimension of the spin j representation is $2j+1$.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

Section A

A1. In quantum mechanics, states of a physical system are associated with elements of a complex vector space with inner product. If v, w are vectors in V, λ is a complex number, and (v, w) is the inner product, how are $(\lambda v, w)$ and $(w, \lambda v)$ related to (w, v) ? **[3]**

A2. The rules for mapping vectors in the vector space V to vectors in the dual space V^* are as follows :

$$|v\rangle \rightarrow \langle v|$$
$$\lambda|v\rangle \rightarrow \langle v|\lambda^*$$
$$A|v\rangle \rightarrow \langle v|A^\dagger$$

where λ is a complex number and A is an operator. Using these rules, show that $\langle v|(A|w\rangle) = (\langle v|A)|w\rangle$ **[3]**

A3. What can we say about the eigenvalues of a hermitian and a unitary operator ? (derivations are not required) **[3]**

A4. What is meant by saying that two vectors v and w are linearly independent ? Determine if the following two vectors are linearly independent ? **[2+2]**

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ , \ w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

A5. Explain why a Stern-Gerlach apparatus uses an *inhomogeneous* magnetic field. How many beams do you expect when particles from a hot gas of spin one particles are passed through a Stern-Gerlach apparatus ? **[2+2]**

A6. A projection operator satisfies $P^\dagger P = P$. Show that this implies P is hermitian. Show also that the eigenvalues of P are either 0 or 1. **[1+2]**

A7. For 3 operators A, B, C write $[A, BC]$ in terms of an expression containing $[A, C]$ and $[A, B]$ **[2]**

A8. Using the property of ladder operators $[a, a^\dagger] = 1$, calculate $[a^2, (a^\dagger)^2]$. **[3]**

A9. If A is a hermitian operator, and θ is a real number, what can you say about the operator $e^{i\theta A}$? Write down the operator which implements, on the Hilbert space of a spin half system, the rotation of the $+z$-direction by an angle β towards the $+y$ axis as an exponential of a Pauli matrix. **[2 + 2]**

A10. If a system of particles transforms in a representation of the angular momentum algebra with dimension 5, what is the eigenvalue of the Casimir J_i^2. **[3]**

Question B1

The Hamiltonian for a simple harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

The *annihilation* and *creation* operators are defined by

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega})$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})$$

(i) Use the canonical commutation relation $[x, p] = i\hbar$ to show that $[a, a^\dagger] = 1$. **[4]**

(ii) The number operator is $N = a^\dagger a$. Express the Hamiltonian in terms of the number operator **[4]**

(iii) If $|n\rangle$ is an eigenstate of the N with eigenvalue n, show that $a^\dagger|n\rangle$ and $a|n\rangle$ are also eigenstates of N, and calculate the corresponding eigenvalues. **[8]**

(iv) Simplify

$$[a, (a^\dagger)^n]$$

and explain what is the energy of the state $[a, (a^\dagger)^n]|0\rangle$ **[4]**

(v) The state $|\lambda\rangle = e^{\lambda a^\dagger}|0\rangle$ is called a coherent state. Simplify $a|\lambda\rangle$ to show that $|\lambda\rangle$ is an eigenstate of a and calculate the eigenvalue. **[4]**

Question B2.

If ψ is a vector in the Hilbert space of a free particle on a line, its components in the basis $|x'\rangle$, denoted by $\psi(x')$, is given by $\psi(x') = \langle x'|\psi\rangle$.

i) Using the matrix elements of the operator p

$$\langle x'|p|x''\rangle = -i\hbar \frac{\partial}{\partial x'}\delta(x' - x'') \tag{1}$$

show that $\langle x'|p|\psi\rangle = -i\hbar\frac{\partial}{\partial x'}\psi(x')$ **[3]**

ii) Consider the state $(1 + \frac{ip\,\delta x'}{\hbar})|\psi\rangle$. Calculate its pairing with the bra $\langle x'|$ and hence explain why p is said to be generator of infinitesimal translations. **[4]**

iii) $|p'\rangle$ is an eigenbasis for the operator p, i.e $p|p'\rangle = p'|p'\rangle$ By considering the matrix element $\langle x'|p|p'\rangle$ explain how to obtain a differential equation for $\langle x'|p'\rangle$. **[6]**

iv) Show that the normalised wavefunction $\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ip'x'}{\hbar}}$ solves the above differential equation **[3]**

v) Write down $\langle p'|x|x'\rangle$ as a function of x'. **[3]**

vi) By using (1) and the completeness relation for continuous eigenstates, prove that $\langle p'|x|p''\rangle = i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')$ **[5]**

Question B3 .

The spin operators for a spin half system are $S_i = \frac{\hbar}{2}\sigma_i$, where the σ_i are the Pauli matrices, and $S_1 = S_x$, $S_2 = S_y$, $S_3 = S_z$. We will denote by $|S_i;+\rangle$ the eigenstate of S_i with positive eigenvalue and $|S_i;-\rangle$ the eigenstate of S_i with negative eigenvalue.

i) Find the eigenvalues and normalized eigenvectors of S_y. Express the normalized eigenvectors as column vectors. Then re-express them as a linear combination in the form $|S_y;\pm\rangle = a|S_z;+\rangle + b|S_z;-\rangle$, where a and b are some numbers which can be read off from the column vectors. **[6]**

ii) A beam of spin half particles in the state $|S_z;+\rangle$ passes through a Stern Gerlach apparatus which measures the spin along the y direction. Explain, by applying the Born Rule, what is probability that the emerging particles are in a state of positive spin ? **[4]**

iii) A beam of spin half particles is prepared in the state

$$|\alpha; t=0\rangle \;=\; \cos(\frac{\beta}{2})\;|S_z;+\rangle \;+\; i\;\sin(\frac{\beta}{2})|S_z;-\rangle$$

and enters, at time $t = 0$, a region with magnetic field, where the Hamiltonian for the spin states is $H = \omega\, S_z$. Explain why the time evolution operator $e^{\frac{-iHt}{\hbar}}$ is unitary. After propagation through the magnetic field for a time t, the state $|\alpha; t=0\rangle$ evolves to $|\alpha;t\rangle$. Calculate $|\alpha;t\rangle$. **[2+ 5]**

iv) For particles in the state $|\alpha, t\rangle$ calculate the probability that the measurement of spin along the y-direction gives $+\frac{\hbar}{2}$. Comment on the answer in the case $\beta = 0$. **[5+2]**

Question B4.

In this question, we will set $\hbar = 1$.

(i) Given the commutation relations $[J_i, J_j] = i\epsilon_{ijk}J_k$, express them in terms of J_3 and the ladder operators $J_\pm = J_1 \pm iJ_2$. **[5]**

(ii) Use the expressions for the orbital angular momentum $L_i = -i\epsilon_{ijk}x_j\frac{\partial}{\partial x_k}$ to prove the commutation relations **[5]**

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

(iii) A particle in a spherically symmetric potential has orbital angular momentum $l = 1$. What is the value of $\vec{L}^2$? What are the possible values of L_3 ? **[5]**

(iv) The $l = 1$ particle above has spin half. The total angular momentum operator is $\vec{J} = \vec{L}+\vec{S}$, where $\vec{S}$ are the spin operators. What are the possible eigenvalues of $\vec{J}^2$ and J_3 ? **[5]**

(v) For the state with maximal values of J and J_3, what are the eigenvalues of L_3, S_3 ? **[2]**

(vi) What is the dimensionality of the space of states with eigenvalues of J_3 equal to $\frac{1}{2}$? **[2]**

End of Examination Paper
Dr S Ramgoolam