BSc/MSci EXAMINATION

PHY-324 Foundations of Quantum Mechanics

Time Allowed: 2 hours 15 minutes

Date: 26-May-2005

Time: 10:00

Instructions: **Answer THREE questions only. Each question carries 20 marks.**

Data: Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Question 1

The spin operators for a spin half system are $S_i = \frac{\hbar}{2}\sigma_i$, with S_1, S_2, S_3 corresponding to spins along the x, y, z directions respectively.

(a) What are the possible outcomes of a measurement of the spin along the x-axis? Derive the corresponding eigenvectors. [**3 marks**]

(b) Consider a unit vector $\hat{n}$ in the $x - z$ plane, in a direction making an angle α with the z-axis. Write the x, y, z components of this vector. [**3 marks**]

(c) Calculate the operator $\vec{S}.\hat{n}$. What are its eigenvalues and normalized eigenvectors ? [**5 marks**]

(d) In a Stern Gerlach experiment, a beam of spin half silver atoms are in an eigenstate of S_z, with eigenvalue $\hbar/2$. They pass through an inhomogeneous magnetic field varying along the direction $\hat{n}$. The outcome is a pair of beams corresponding to the two allowed values of $\vec{S}.\hat{n}$. What is the ratio of the intensities of the two beams ? [**6 marks**]

(e) Suppose the beam corresponding to the positive eigenvalue of $\vec{S}.\hat{n}$ above is passed through another Stern-Gerlach apparatus measuring the spin along the z-axis. What is the probability that the outgoing atoms have spin $\hbar/2$ along the z-axis ? [**3 marks**]

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Question 2

A spin 1/2 system is governed by the Hamiltonian

$$H = \omega S_z \ ,$$

where $S_z = \hbar \frac{\sigma_3}{2}$, and μ is a constant.

(a) Compute the energy eigenvalues of the system and the corresponding eigenvectors. **[2 marks]**

(b) Suppose the system is initially in an eigenstate of S_y with eigenvalue $\hbar/2$ at time $t_0 = 0$. What is the normalized state of the system at a time $t > 0$? **[4 marks]**

(c) Calculate the expectation values of $\langle S_y \rangle$, $\langle S_x \rangle$ at a time t. **[4 marks]**

(d) What is the first time $t > 0$ when the system is in the same eigenstate it started with ? Explain how this result is consistent with your answer to part 3 above. **[4 marks]**

(e) Calculate the dispersions $\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - (\langle S_x \rangle)^2$ and $\langle (\Delta S_y)^2 \rangle = \langle S_y^2 \rangle - \langle S_y \rangle^2$ at time $t > 0$. Explain why your results are consistent with the uncertainty principle. **[6 marks]**

(*Hint* : $\langle\, (\Delta A)^2 \,\rangle \ \langle\, (\Delta B)^2 \,\rangle \geq \ \frac{1}{4} \ |\, \langle\, [A, B] \,\rangle \,|^2$)

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Question 3

A particle of mass $m = 1$ is subject to the one-dimensional harmonic oscillator potential with frequency $\omega = 1$, so that the system is governed by the Hamiltonian

$$H_0 = \frac{p^2}{2} + \frac{x^2}{2} \, .$$

(a). Derive the form of the Hamiltonian in terms of the creation operator $a^\dagger = \frac{x+ip}{\sqrt{2\hbar}}$ and the annihilation operator $a = \frac{x-ip}{\sqrt{2\hbar}}$ **[4 marks]**

(b) The state vector at initial time $t_0 = 0$ is a linear combination of the ground state and the first excited state.

$$|\alpha, t_0 = 0 > \;\; = \;\; \frac{1}{\sqrt{2}}(\;\; |0> + i|1> \;\;)$$

Calculate the state vector $|\alpha, t_0 = 0; t>$ of the system at a generic time $t > 0$. **[5 marks]**

(c) Determine the expectation values $\langle t| \, x \, |t\rangle$ of the position operator at a generic time $t > 0$. **[4 marks]**

(d) Consider the system described by the Hamiltonian $H = H_0 + V$, where H_0 is the original hamiltonian, and V is the small perturbing potential

$$V(x) = \beta \, x^4 \, ,$$

where β is a constant. Calculate the correction to the eigenvalue of the energy for the ground state, to the first order in the parameter β. **[7 marks]**

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Question 4

(a) The pions π^+, π^-, π^0 form an irreducible representation of $SU(2)$ isospin symmetry. What is the total isospin I of this set of particles ? What are the I_3 eigenvalues of these particles ? [**4 marks**]

(b) The proton and neutron can be viewed as two states of the nucleon in an $I = \frac{1}{2}$ represetation of isospin symmetry. They can be viewed as states $|I = \frac{1}{2}, I_3 = \frac{1}{2} >$ and $|I = \frac{1}{2}, I_3 = \frac{-1}{2} >$ respectively. A two-particle composite system made of proton and neutron can decomposed into eigenstates of total isospin, denoted $I^{(t)}$ and total $I_3^{(t)}$. Decompose states labelled by $I^{(t)}$ and $I_3^{(t)}$ in terms of $I^{(1)}, I_3^{(1)}$ and $I^{(2)}, I_3^{(2)}$ of particle 1 and 2 respectively. [**8 marks**]

(c) The deuteron is in fact a composite particle of the type considered above with $I^{(t)} = I_3^{(t)} = 0$. Explain using this information, and assuming isospin symmetry of the interactions, why the processes below

$$I : p + p \rightarrow d + \pi^+$$
$$II : p + n \rightarrow d + \pi^0$$

have the property that the ratio of their cross-sections $\frac{\sigma(II)}{\sigma(I)}$ is approximately $\frac{1}{2}$. [**8 marks**]

Question 5

(a) Three particles have spins $\frac{1}{2}, 1, 2$ respectively. What are the allowed eigenvalues of $J_1^2 + J_2^2 + J_3^2$, where $\vec{J}$ is the total angular momentum operator ? For each eigenvalue, what is the multiplicity of states ? **[7 marks]**

Hint The formulas $j(j+1)$ for the Casimir and $(2j+1)$ for the dimension of a spin j multiplet will be useful.

(b) Express the commutation relations $[L_i, L_j] = i\epsilon_{ijk}L_k$ in terms of the diagonal operator L_3 and the ladder operators $L_\pm = L_1 \pm iL_2$. **[6 marks]**

(c) Use the coordinate space realization $L_i = -i\epsilon_{ijk}x_j\partial_k$ to show that for $x_+ = x + iy$, $L_+(x_+) = 0$. Calculate $L_-(x_+), L_-^2(x_+), L_-^3(x_+)$. Calculate the action of the operator $L_1^2 + L_2^2 + L_3^2$ on x_+ and comment on the answer you got for $L_-^3(x_+)$ in the light of the representation of the angular momentum algebra that x_+ belongs to. **[7 marks]**

End of Examination Paper **Dr S. Ramgoolam**