Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-218 Mathematical Techniques 3

Time Allowed: 2 hours 15 minutes

Date: 14 May 2007

Time: 14.30

Answer ALL questions in Section A. Attempt TWO questions from section B.

Section A carries 40 marks, the questions in Section B each carry 30 marks. Indicative marks for questions are given by [ ] in bold face.

The use of numeric calculators is permitted.

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

Section A

A1 Check whether the following pairs of vectors are linearly independent or not

$$\mathbf{u} = (1,-4,2)\ ,\ \ \mathbf{v} = (-2,8,-4)$$ [4]

A2 A function $g(x)$ is an eigenfunction of the linear differential operator L with corresponding eigenvalue λ. Express this statement as an equation.
Verify that the polynomial $g(x) = 8x^3 - 12x$ is an eigenfunction of the operator

$$L = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}\ .$$

What is its eigenvalue? [5]

A3 Solve the 1st order differential equation

$$\frac{dy}{dx} = \frac{x^3}{y}$$

subject to the boundary condition that $y(x=0) = 1$ [4]

A4 Solve the 2nd order differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$ [6]

with the boundary condition $P(x,t=0) = 1$.

A5 Solve the partial differential equation

$$U(x,t)\frac{\partial U(x,t)}{\partial x} = \frac{\partial U(x,t)}{\partial t}$$ [5]

by using the method of separation of variables, so that
$U(x,t) = f(x)g(t)$ where $f(x)$ and $g(t)$ are functions to be determined.

A6 Give a definition of the dirac delta function $\delta(x-a)$ where a is a constant. [2]

Calculate the following integrals:

Question continues over page

(i) $\int_0^1 (4x^2 - 1)\delta(x - \frac{1}{2})dx$ (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} y^2 \delta(x)\delta(y - 2\pi)dxdy$

[4]

Section B

B1 (a) Give the definition of the inner product, $\langle f, g \rangle$, between two functions $f(x)$ and $g(x)$, show that the inner product satisfies the following properties

(i) $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$

(ii) $\langle f, g \rangle = \langle g, f \rangle$

(iii) $\langle f, ag \rangle = a\langle f, g \rangle$ where a is a real number [6]

(b) Let $f(x), g(x)$ be two square integrable functions. Given the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

write down an equation that defines the adjoint L^+ of the linear operator L
Calculate the adjoint operator L^+ when $L = \frac{d^2}{dx^2}$.

[9]

(c) The first three Legendre Polynomials are given by

$$P_0(x) = 1 \ , \quad P_1(x) = x \ , \quad P_2(x) = 3x^2 - 1$$

show that P_0, P_1 and P_3 are mutually orthogonal. You may assume the inner product between Legendre Polynomials $P_i(x)$ and $P_j(x)$ is

$$\langle P_i, P_j \rangle = \int_{-1}^{1} P_i(x)P_j(x)dx$$ [10]

B2 (a) Solve the following differential equations

(i) $\frac{dy}{dx} = \sin(x)$ with $y = 0$ at $x = \frac{\pi}{2}$

(ii) $y\frac{dy}{dx} = \frac{x}{x^2 + 1}$ with $y = 1$ at $x = 0$ [8]

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(b) Solve the simultaneous differential equations

$$\frac{df(t)}{dt} = f(t) \ , \quad \frac{dg(t)}{dt} = f^2(t)$$ [8]

(c) The current in a resistance -inductance circuit driven by an exponentially increasing voltage $V(t) = e^{\gamma t}$ satisfies the following differential equation

$$L\frac{dI(t)}{dt} + RI(t) = e^{\gamma t}$$

where R is the resistance and L the value of the inductance.
Find an expression for the current $I(t)$. [9]

B3 The equation governing heat flow in 1-space dimension is

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{K}\frac{\partial T(x,t)}{\partial t}$$

where the function $T(x,t)$ defines the temperature at point x at time t and $K > 0$ is a constant.

(a) Obtain a general solution for $T(x,t)$ assuming separation of variables. [7]

(b) A metal rod of length L is placed along the x-axis so its ends are located at $x = 0$ and $x = L$ respectively. At time $t = 0$, $T(x,0) = T\delta(x - L/2)$, where T is a constant. Given the boundary conditions $\frac{\partial T(0,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x} = 0$ find an expression for the temperature $T(x,t)$. [9]

(c) Evaluate $T(x,t)$ in the limit as time $t \to \infty$ [9]

You may use the following identities

$$\frac{1}{L}\int_{-L}^{L}\cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{L})dx = \frac{2}{L}\int_{0}^{L}\cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{L})dx = \delta_{nm}$$

for $n = 0,1,2,3,..$

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B4 The Green's function $G(x, y)$ of the Laplacian operator in 2-d satisfies the differential equation

$$\nabla^2 G(x,y) = \delta(x)\delta(y)$$

where $\nabla^2 = \overline{\nabla} \cdot \overline{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator in 2-d.

(a) Show that $G(x,y) = \frac{1}{2\pi} \ln r$ with $r = \sqrt{x^2 + y^2}$. **[12]**

You may assume the Gauss theorem in 2-d:-

$$\int_A (\overline{\nabla} \cdot \overline{D}) dA = \int_C \overline{D} \cdot d\overline{S}$$

with $\overline{D}$ any 2-d vector and A is any 2-d region whose boundary is C.

(b) Flatland is a world with only 2 spatial dimensions. Two charges are placed along the x-axis, $-Q$ at $x = x_1$ and $+Q$ at $x = -x_1$. The electric potential $V(x,y)$ at a given point in Flatland, satisfies Poisson's equation:

$$\nabla^2 V(x,y) = -\frac{\rho(x,y)}{\varepsilon_0}$$

with $\rho(x,y)$ the charge density.

Write down the expression for $\rho(x,y)$ corresponding to the two charge configuration discussed above. **[5]**

Hence calculate the expression for the potential $V(x,y)$. **[8]**

End of Paper **Dr O.A. Soloviev**