Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-217 Vibrations and Waves

Time Allowed: 2 hours 15 minutes

Date: 03 May 2007

Time: 10:00

Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 40 marks, each question in section B carries 30 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

NUMERIC CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

Mathematical formula

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

Version 3

Section A

A1 A string of total length L is fixed at its ends.

(a) Sketch the three lowest modes of transverse oscillation. [3]

(b) How many normal modes of transverse oscillation does an ideal string have? [1]

A2 A travelling wave $y(x,t)$ has a wavenumber k, angular frequency ω and moves to the right along the x-axis.

(a) Write down an expression for $y(x,t)$. [1]

(b) Sketch graphs of $y(x,t)$ both at a fixed point $x = 0$ on the x-axis, and at a fixed time $t = 0$. [2]

(c) Indicate the wavelength λ and the period T on the appropriate graph. [2]

A3 A spring with spring constant s is suspended vertically. A mass $m = 1.2$ kg is attached to its free end and at equilibrium, the spring extends by $\Delta = 0.05$ m. The mass is then pulled down a further 0.15 m, released from rest at time $t = 0$ and found to execute simple harmonic motion.
($g = 9.81$ m s^{-2}).

(a) (i) Obtain the equation of motion. [2]

(ii) Show that the displacement $x(t)$ from its resting position has the general form

$$x(t) = A_0 \cos(\omega_0 t + \varphi).$$ [4]

(b) Using the above data, calculate the numerical values of the following

(i) the amplitude and the constant phase angle; [3]
(ii) the frequency and the period; [3]
(iii) the maximum and minimum speed of the mass; [2]
(iv) the maximum kinetic energy, and hence, giving an argument based on the conservation of energy, the total energy E. [2]

A4 The displacement $x(t)$ from Question A3 is a solution of the simple harmonic oscillator equation and can be obtained from the complex variable $z(t)$ satisfying the same equation of motion.

(a) Write down this equation and the mathematical form of its general complex solution $z(t)$. [2]

(b) The variable $z(t)$ can be depicted in an Argand (or phasor) diagram as a rotating vector. Draw an Argand diagram showing this vector at arbitrary time t, indicating its magnitude, the phase angle of the vector, and the sense and magnitude of its angular velocity. Also indicate the physical displacement $x(t)$. [3]

Please turn to next page

A5 For the oscillator discussed in Question A3, sketch $x(t)$. Indicate on your sketch the amplitude and the period. [2]

A6 A general equation for the phase velocity $v(\lambda)$ of a surface wave on a liquid is

$$v^2(\lambda)=\left[\frac{g\lambda}{2\pi}+\frac{2\pi\sigma}{\lambda\rho}\right]\tanh\left(\frac{2\pi h}{\lambda}\right),$$

where λ = wavelength, σ = surface tension, ρ = density of the liquid, and h is the depth of the liquid. Find the formula for λ_{equal} where the gravity and surface tension terms are equal. [2]

A7 (a) Discuss the difference between transverse and longitudinal waves. [2]

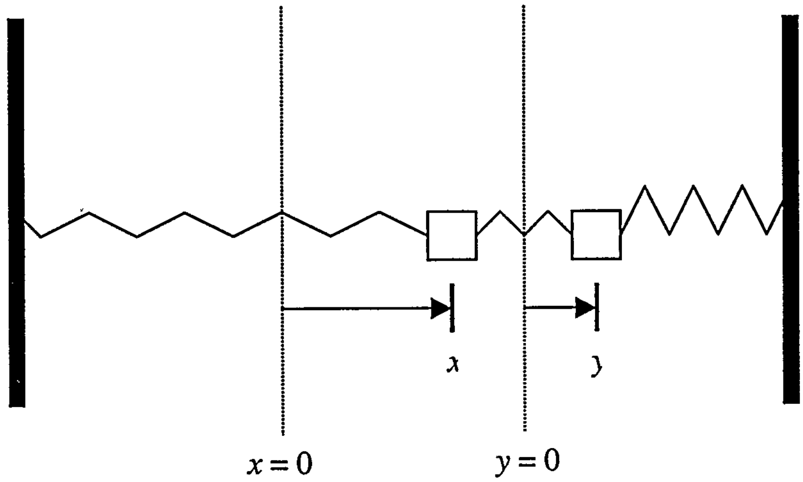
(b) Does a sound wave carry energy, mass, or both? [1]

(c) State whether transverse and/or longitudinal waves can propagate through solids, liquids, and gasses. [3]

Please turn to next page

Section B

B1 (a) Two equal masses m, are attached to two opposing walls by two identical springs of spring constant $2s$ and coupled by a third spring of spring constant s (as illustrated below).



(i) Show that the equations of motion of the two masses are

$$m\ddot{x} = -3sx + sy$$
$$m\ddot{y} = -3sy + sx.$$ [6]

(ii) Determine the normal mode angular frequencies by seeking solutions of the form

$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t}.$$ [2]

(iii) Show that the normal mode angular frequencies are in the ratio $\omega_1 : \omega_2 = 1 : \sqrt{2}$. [6]

(b) (i) Explain how to excite the system to vibrate in each of the two normal modes. [3]

(ii) Describe the motion of the masses in the two normal modes. [1]

(iii) Explain why the low-frequency mode is independent of the coupling spring. [3]

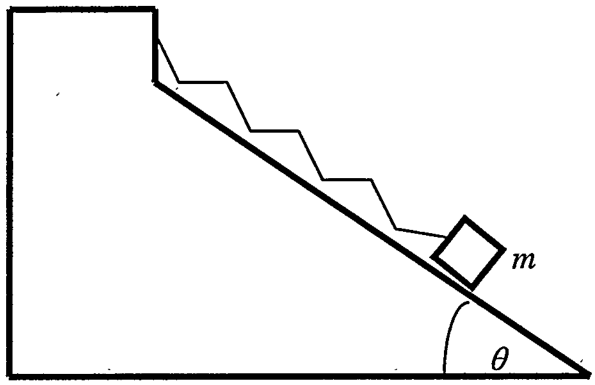
(c) Imagine the left mass is initially held in its equilibrium position ($x = 0$) while the other mass is moved some amount y_0 to the right. The masses are then releases from rest.

(i) Describe the salient features of the subsequent motion with particular reference to the exchange of energy between the two masses. [6]

(ii) Draw sketches of $x(t)$ and $y(t)$ to illustrate your description of the exchange process. Show in your sketches the time it takes for the right mass to come to rest. [3]

Please turn to next page

B2 (a) A mass m is attached to a massless spring with a spring constant s. The mass can slide along a frictionless plane inclined as an angle θ to the ground (as illustrated below).



Show that the period of the oscillation is independent of the angle θ. [10]

(b) Imagine an idealised hole drilled through the Earth connecting the north and south poles. Assume that the Earth is a perfect sphere of radius 6370 km and is of constant density, and that the hole contains a perfect vacuum. An object is dropped down the hole.

(i) Show that the motion of the object is simple harmonic. [4]

(ii) By explicit consideration of the boundary conditions, calculate the amplitude of the motion. Fully explain your method. [8]

(iii) Determine an equation for the displacement of the object from the centre of the Earth as a function of time, $r(t)$. [2]

(iv) Determine the period of oscillation of the object. [6]

Note that Newton's Law of Universal Gravitation shows that the gravitational field at a radius r inside a spherical distribution of matter is only due to the matter inside the sphere of radius r.

Thus, $F_r = -\frac{Gm_o M_r}{r^2}$,

where $M_r = (4/3\pi\rho)r^3$ (p is the density of the Earth, m_o is the mass of the object, r is the distance from the centre of the Earth, and G is the universal gravitation constant). At the surface of the Earth, $g = 9.81$ m s^{-2}.

Please turn to next page

B3 Consider a string with mass per unit length μ stretched under tension T between two fixed points a distance L apart. The speed of a wave travelling along the string is

$$v = \sqrt{T/\mu}.$$

The transverse displacement of the string obeys the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v}\frac{\partial^2 y}{\partial t^2} = 0.$$

(a) By assuming the string vibrates harmonically, with a frequency ω according to the relation

$$y(x, y) = A(x)\sin(\omega t),$$

solve for $A(x)$ and find the normal mode frequencies ω_n. [15]

(b) By considering the first three normal modes of a string explain why the normal mode wavenumber $k_n = 2\pi/\lambda_n$ is proportional to the mode number n. [6]

(c) The E-string of a violin has the fundamental frequency ν_1 = 640 Hz, a length L = 0.330 m and a mass per unit length μ = 0.375 x 10^{-3} kg m^{-1}. Calculate the tension T in the string and the velocity of a wave travelling along the string. [6]

(d) Suppose a violinist firmly places a finger halfway along the string whilst playing the instrument. What is the frequency of the lowest note heard? Explain your reasoning. [3]

B4 (a) Derive the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v}\frac{\partial^2 y}{\partial t^2} = 0$$

for a small displacement $y(x,t)$ of a string of mass per unit length m under tension T, and show that the speed of a wave travelling along the string is

$$v = \sqrt{T/\mu}.$$ [8]

(b) Two equal amplitude harmonic waves are superimposed.

$$y_1(x,t) = A\cos(k_1 x - \omega_1 t) \text{ and } y_2(x,t) = A\cos(k_2 x - \omega_2 t)$$

(i) Show that the resultant wave has the form

$$y_1(x,t) = 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos(\bar{k}x - \bar{\omega}t)$$ [3]

Question continues over page

(ii) Sketch the wave at $t = 0$ and small Δk and $\Delta \omega$. [3]

(iii) Hence provide general definitions of phase velocity and group velocity for the superposition of waves of similar frequencies. [5]

(c) Use your sketch from Question B4(b)(ii) to show that wave groups of size Δx satisfy the bandwidth theorem

$$\Delta x \Delta k \approx 2\pi$$ [4]

(d) Use the theorem to discuss the special extent and wavelength composition of

(i) a monochromatic wave, and [2]
(ii) a localised wave packet or pulse [2]

(e) In quantum physics, a free particle of mass m, momentum $p = \hbar k$, and energy $E = p^2/2m$, obeys the dispersion relation

$$\omega = \frac{\hbar}{2m} k^2$$

Show that the group velocity is twice the phase velocity. [3]

End of Examination Paper

Dr T J S Dennis

