Queen Mary
University of London

BSc EXAMINATION

PHY-216 Mathematical Techniques II

Time Allowed: 2 hours 15 minutes

Date: 23rd May 2005
Time: 14:30

Answer ALL questions. All four questions carry equal marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.

You may use any calculator.

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

THIS PAGE TO BE LEFT BLANK

1. (a) Draw an Argand diagram and mark the complex number $z = a + i\,b$ on it. If z is expressed in polar form $z = r\,e^{i\theta}$, given expressions for a and b in terms of r and θ. **[5]**

   (b) On an Argand diagram, mark the four fourth roots of $e^{i\pi}$. **[3]**

   (c) Write down each of the four fourth roots of $e^{i\pi}$ in the form $a + i\,b$ and also in the form $r\,e^{i\theta}$. **[6]**

   (d) Write down an expression for the scalar product **a.b** of two vectors, $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$, in terms of their components. **[4]**

   (e) Write down an expression for the vector product **a×b** of two vectors, $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$, in terms of their components. Hence find the cross products of the unit vectors, $\hat{i} \times \hat{j}$, $\hat{j} \times \hat{k}$, $\hat{i} \times \hat{k}$. **[7]**

2. (a) Let h be the scalar field $h = x^2 + y^2 + z^2$. Find ∇h, the gradient of h. Express your answer in terms of $\hat{\mathbf{r}}$, the unit vector in the radial direction, and r, the magnitude of the radius vector $\mathbf{r} = (x, y, z)$. **[8]**

   (b) Write down expressions for div $\mathbf{A}(x, y, z)$ and curl $\mathbf{A}(x, y, z)$ where $\mathbf{A}$ is a vector field. Evaluate these expressions for the cases where $\mathbf{A} = r^2\,\mathbf{r}$. **[8]**

   (c) Show that div curl $\mathbf{A}$ vanishes, where $\mathbf{A}$ is any vector field. **[9]**

Please turn to the next page

3. (a) For the matrices $\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ show that **AB** is not the same as BA, and that $\mathbf{A}^T\mathbf{B}^T = (\mathbf{BA})^T$. **[5]**

(b) Define an a*symmetric* matrix, an *orthogonal* matrix, a *unitary* matrix and a *Hermitian* matrix, and give a 2×2 numerical example of each with real or complex elements as appropriate. **[5]**

(c) **M** is the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$. Find the eigenvalues and eigenvectors of **M**.

[10]

(d) Compare the eigenvalues you obtain with the trace and with the determinant of **M**. **[5]**

4. (a) Evaluate the integral $\int_0^4 \int_0^3 (x^2 + xy)\, dxdy$. **[5]**

(b) A function $f(t)$ is periodic on t with a period of 2π, and is given by

$$f(t) = \begin{cases} 1 & \text{for } t = 0 \text{ to } \pi \\ 0 & \text{for } t = \pi \text{ to } 2\pi \end{cases}$$

Sketch the function, and calculate the Fourier coefficients a_k and b_k of $f(t)$ for $k = 0$ to 4. By inspection of these coefficients, give an expression for the Fourier expansion of $f(t)$ valid for $k = 0$ to infinity. **[10]**

(c) Given a 2nd-order linear differential equation,

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(x)$$

$y_p(x)$ is a particular solution and $y_c(x)$ is a complementary function. Show that $y_p(x) + y_c(x)$ is a general solution and indicate where the constants of integration will appear. **[10]**

End of Examination Paper
Prof. D.J. Dunstan