Queen Mary
University of London

BSc/MSci EXAMINATION

PHY213 Space, Time and Gravity

Time Allowed: 2 hours 15 minutes

Date: 20th May 2005

Time: 10:00

Candidates should answer FIVE of the eight questions in Section A, each of which carries 8 marks, and TWO of the four questions in Section B, each of which carries 30 marks; SEVEN questions should be answered altogether. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A numeric calculator is permitted.

Notation:

Spatial co-ordinates denoted by the lower-case letters x, r etc. have the dimensions of length whilst those denoted by the upper case letters X, R etc. are measured in light-travel time. The dimensionless velocity β is defined by

$$\beta = \frac{dX}{dt} \quad \text{or} \quad \beta = \frac{dR}{dt}, \text{etc}$$

The Lorentz transformations:

The Lorentz transformation between the frame (t,X) of an inertial observer O and the frame (t',X') of an observer O′, moving at constant velocity β_o in the positive X-direction with respect to O, is given by

$$t' = \gamma_o[t - \beta_o X]; \quad X' = \gamma_o[X - \beta_o t] \quad \text{where} \quad \gamma_o = 1/\sqrt{1-\beta_o^2}$$

Data

Gravitational constant	G	6.67×10^{-11}	N m^2 kg^{-2}
Speed of light	c	3.00×10^{8}	m s^{-1}
Mass of sun	M_{sun}	1.99×10^{30}	kg
Radius of sun	R_{sun}	6.96×10^{8}	m
Distance of earth from sun	1 AU	1.50×10^{11}	m
Year	1 y	3.16×10^{7}	s

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[Answer 5 questions. Each question carries 8 marks.]

A.1 Give one reason why Galilean space-time is inadequate for describing the real world. **[4 marks]**

With the help of a diagram, show how an observer assigns a time to an event that is not on his world-line. **[4 marks]**

A.2

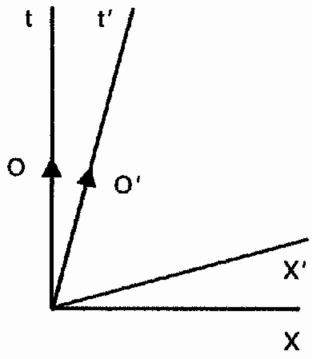

Figure 1.

Figure 1 shows in a Minkowski diagram the frames of reference of two observers, O and O′, who is moving in the positive *X*-direction with respect to O. Explain **briefly** why, although its axes are orthogonal in this diagram, the frame of O has no preferred status. **[4 marks]**

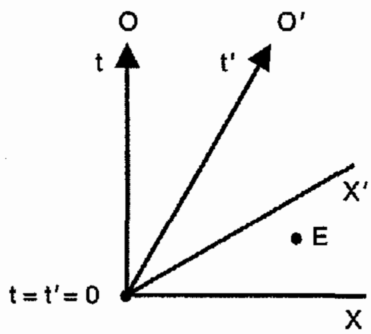

Figure 2.

Observers O and O′ are at the origin of their rest-frames shown in Figure 2. At $t = t' = 0$, event E is in the past as far as O′ is concerned but is in the future of O. Explain **briefly** why O cannot influence the event E. **[4 marks]**

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A.3 The four-velocity of a particle with velocity β and Lorentz factor γ in some frame is given by $\mathbf{v} = c\begin{bmatrix} \gamma\, dt/dt \\ \gamma\, dX/dt \end{bmatrix} = c\gamma \begin{bmatrix} 1 \\ \beta \end{bmatrix}$.

Show that the magnitude of the four-velocity is c. **[4 marks]**

Consider the following definition of a "four-velocity" of a particle, in some frame,

$$v := \frac{d\mathbf{\rho}}{dt},$$

where $\mathbf{\rho}$ is the particle's four-displacement and t is the time co-ordinate in that frame.

Show that this does not define a four vector.

[4 marks]

A.4 Sketch, in an appropriate Minkowskian frame, the world-line of a particle undergoing constant proper acceleration (hyperbolic motion). **[2 marks]**

Indicate on a Minkowski diagram the particle's past and future *horizons* **[2 marks]** and explain why they are so called. **[4 marks]**

A.5 State *Hubble's law* **[2 marks]**. What observations were incompatible with Hubble's original value of H_0? **[2 marks]**

Draw a set of world-lines for fundamental Milne observers in a *Minkowskian* set of co-ordinates. **[2 marks]** Indicate the limits between which these world-lines must lie. **[2 marks]**

A.6 State the *equivalence principle*. **[3 marks]**

With the help of a diagram, show that the principle holds only *locally*. **[5 marks]**

A.7 It has been said that "matter tells space-time how to curve and space-time tells matter how to move". Rewrite this in the language of relativity. **[4 marks]**

Einstein's field equations are

$$G_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu}.$$

What do the quantities $G_{\mu\nu}$ and $T_{\mu\nu}$ represent? **[4 marks]**

A.8 With the help of a diagram, explain what is meant by the relativistic *precession of perihelion* of the planet Mercury. **[4 marks]**

Sketch the positions of the regions I, II, III and IV of Schwarzschild space-time in a **Penrose** diagram. **[4 marks]**

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Section B (Answer 2 questions. Each question carries 30 marks.)

B.1 Use the concept of planes of simultaneity for relatively moving observers to explain **qualitatively** the so-called twin paradox. **[10 marks]**

An inertial observer O′ travels in the positive *X*-direction with velocity β_0 with respect to another inertial observer O. A particle travels in the positive *X*-direction with velocity β with respect to O. Show that O′ determines the particle's velocity to be β' given by

$$\beta' = \frac{\beta - \beta_0}{1 - \beta_0 \beta}.$$ **[5 marks]**

Deduce that the velocity of a photon is the same in all reference frames. **[3 marks]**

An interstellar spaceship travels at 90% of the speed of light, relative to the earth, towards a star that is 6×10^{16} m from earth. When it is halfway to the star, the spaceship launches a probe towards the star at 50% of the speed of light relative to itself. What is the speed, expressed as a percentage of the velocity of light, of the probe as observed from earth. **[4 marks]**

How long does the probe take to reach the star, from the point at which it was launched, as determined by (a) an observer on earth **[2 marks]**, (b) according to the probe's clock **[6 marks]**?

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B.2 The four-velocity $\mathbf{\upsilon}$ of any particle has magnitude c. Sketch the locus of all possible four-velocities in the (υ_t, υ_X) plane, where υ_t and υ_X are the time and space components of $\mathbf{\upsilon}$ in some frame. **[5 marks]**

The four-momentum $\boldsymbol{\pi}$ of a particle of rest-mass m_0 is defined as

$$\boldsymbol{\pi} = m_0 \mathbf{\upsilon}$$

and can be written in terms of its components in some frame as

$$\boldsymbol{\pi} = \begin{bmatrix} E/c \\ p \end{bmatrix},$$

where E is the relativistic energy of the particle and p its relativistic momentum. Show that

$$E^2 - p^2c^2 = m_0^2c^4 . \qquad \textbf{[5 marks]}$$

What is the momentum of a photon in terms of its energy? **[1 mark]**

The relativistic energy is also given by

$$E = \gamma m_0 c^2 ,$$

where γ is the Lorentz factor of the particle. Deduce that the relativistic kinetic energy E_{kinetic} is given by

$$E_{\text{kinteic}} = (\gamma - 1) m_0 c^2 . \qquad \textbf{[3 marks]}$$

What fraction of the relativistic energy of a particle travelling at 99% of the velocity of light is contributed by its rest-mass? **[3 marks]** How fast does the particle have to be travelling for its rest mass to be 1% of its relativistic energy? **[3 marks]**

Show that, for velocities much less than that of light,

$$E \approx \left[m_0 c^2 + \frac{1}{2} m_0 v^2 \right],$$

v being the ordinary velocity of the particle. **[5 marks]**

Is the previous equation accurate to better than 1% for a particle travelling at 100 km s^{-1}? **[2 marks]** Calculate the ratio of the kinetic to the rest-mass energy of a particle travelling at 1 km s^{-1}. **[3 marks]**

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B.3 State the *cosmological principle*. **[3 marks]**

Minkowskian co-ordinates (t,R) are related to Milne co-ordinates (τ,χ) by the transformations

$$t = \tau\sqrt{1+\chi^2};$$
$$R = \chi\tau.$$

Show that τ is related to t by

$$\tau = t\sqrt{1-\frac{R^2}{t^2}}.$$ **[4 marks]**

Hence, using Lorentz time-dilation, show that τ is the proper time of the fundamental observer at event (t,R) **[4 marks]**

A spatial section at constant Minkowskian time t was described by Milne as *a private space* whereas a section at constant Milne time τ he called a *public space*. Explain this terminology. **[5 marks]**

Starting from the form of the line-element in Minkowskian co-ordinates, show that its form in Milne co-ordinates is

$$\frac{ds^2}{c^2} = d\tau^2 - \tau^2\frac{d\chi^2}{\left(1+\chi^2\right)}.$$ **[6 marks]**

Show that the proper distance $\ell(\tau)$ from the origin at time τ of a galaxy with co-ordinate χ is given by

$$\ell(\tau) = c\tau\int_0^{\chi}\frac{d\chi'}{\sqrt{1+\chi'^2}}.$$ **[4 marks]**

and that the rate of change $\dot{\ell}(\tau)$ of proper distance with τ is given

$$\dot{\ell}(\tau) = c\int_0^{\chi}\frac{d\chi'}{\sqrt{1+\chi'^2}}.$$ **[1 mark]**

Derive an expression for the Hubble parameter as a function of τ. **[3 marks]**

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B.4 The Schwarzschild line-element for a spherically symmetric gravitating object of mass M is given by

$$ds^2 = \left(1-\frac{2\mu}{\hat{r}}\right)c^2 d\hat{t}^2 - \left[d\hat{r}^2 \Big/ \left(1-\frac{2\mu}{\hat{r}}\right) + \hat{r}^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right],$$

where

$$\mu = GM/c^2 .$$

What is the surface area of a sphere with co-ordinate radius $\hat{r}$? **[3 marks]**

Show that the proper radial ℓ_{12} distance between points with co-ordinate radii $\hat{r}_1$ and $\hat{r}_2$, both greater than 2μ is given by

$$\ell_{12} = \int_{\hat{r}_1}^{\hat{r}_2} \frac{d\hat{r}}{\left(1-\frac{2\mu}{\hat{r}}\right)^{1/2}} .$$ **[5 marks]**

Say, with explanation, whether this distance is greater than, or less than, the corresponding distance between r1 and r2 in a flat space. **[4 marks]**

Show that the equation of the gradient of the world-line of a radially inward travelling photon is

$$\frac{d\hat{t}}{d\hat{r}} = -\frac{1}{c\left(1-\frac{2\mu}{\hat{r}}\right)}$$ **[5 marks]**

Sketch this world-line from a finite $\hat{r}$ down to near $\hat{r} = 2\mu$. **[3 marks]**

The gravitational redshift z of a photon emitted at $\hat{r}_e > 2\mu$ and received at $\hat{r}_r > \hat{r}_e$ is given by

$$1+z = \left(1-\frac{2\mu}{\hat{r}_r}\right) \Big/ \left(1-\frac{2\mu}{\hat{r}_e}\right).$$

Estimate the redshift of a photon emitted at the sun's surface and received at the earth. **[5 marks]**

Discuss briefly why Schwarzschild co-ordinates are unsuitable for representing the structure of Schwarzschild space-time within the horizon. **[5 marks]**

End of Examination Paper Dr WJ Spence

