Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-121 Mathematical Techniques 1

Time Allowed: 2 hours 15 minutes

Date: 12th May 2006

Time: 14:30

Answer all of section A and three questions from section B. An indicative marking scheme is shown in square brackets [ ] after each part of a question.

You may use an electronic calculator during this examination.
You may retain the question paper at the end of the examination.

Useful formulae:

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).$$

$$< y >= \frac{1}{b-a}\int_a^b y\,dx$$

$$V = \pi\int_a^b y^2 dx \qquad \bar{x}\int y\,dx = \int xy\,dx$$

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \text{ and } \tan\left(\frac{\pi}{4}\right) = 1$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \text{ and } \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

Section A: Answer all questions in this section.

1. **a)** Differentiate the following with respect to x:

i) $(ax+b)^m$ **ii)** $\sin^2(x)$ **iii)** $e^{\alpha x}$ **iv)** $e^x \sin(x)$ **v)** $\frac{\sin(x)}{x}$ **[5]**

b) Find all the first and second partial derivatives of $z = 3x^2 + 2y^3$. **[5]**

c) i) Write the first four terms of a Maclaurin series expansion for $f(x)$.

ii) Write the first four terms of a Taylor series expansion for $f(x+h)$. **[2]**

2. Evaluate all the solutions in the interval $0 \le x \le 2\pi$ for the following:

a) $x = \arcsin\left(\frac{1}{2}\right)$ **b)** $x = \arccos\left(\frac{-\sqrt{3}}{2}\right)$ **c)** $x = \arctan(-1)$ **[3]**

3. A curve is defined parametrically as $x = 4t$ and $y = 2t^2$.

a) Calculate $\frac{dy}{dt}$. **b)** Calculate $\frac{dx}{dt}$ and $\frac{dt}{dx}$, hence calculate $\frac{dy}{dx}$. **[2]**

c) Calculate $\frac{d\left(\frac{dy}{dx}\right)}{dt}$ and consequently $\frac{d^2y}{dx^2}$. **[2]**

4. **a)** Express the following in the form $a+bi$:

i) $(2+i)-(4-3i)$ **ii)** $(2+i)(4-3i)$ **iii)** $\frac{(2+i)}{(4-3i)}$ **iv)** $2e^{\frac{\pi}{3}i}$ **[5]**

b) Express $\sqrt{3}+i$ in complex exponential form. **[2]**

5. **a)** Write down or derive the following integrals:

i) $\int 2x^3 dx$ **ii)** $\int \sin(\pi x) dx$ **iii)** $\int e^{3x}\, dx$ **iv)** $\int xe^{-x} dx$ **[5]**

b) Evaluate the following multiple integral: $\int_0^1 \int_0^1 x^2 y \, dx\, dy$ **[2]**

6. A lamina is formed by the curve $y = 1 - x^2$, the x-axis and the y-axis, in the positive quadrant (i.e. between $x = 0$ and $x = 1$).

a) Calculate the average y value in this interval (the average height). **[2]**

b) Calculate the x-coordinate of the centroid of the lamina. **[5]**

c) Calculate the volume of the solid formed when the lamina is rotated about the x- axis. **[5]**

Total Section A [40]

Section B: Answer <u>three</u> questions from this section.

7. **a)** Given that $\mathrm{Re}^{i\theta} = R(\cos\theta + i\sin\theta)$, derive exponential expressions for $\cos\theta$ and $\sin\theta$.

b) For a satellite in circular orbit around a planet of mass, m, the angular frequency, ω, is related to the orbital radius, r, by: $\omega^2 = \dfrac{Gm}{r^3}$, where G is the gravitational constant. Find an expression for the frequency change, δω, due to small changes in the variables, δm and δr. Hence evaluate the fractional change in ω, $\dfrac{\delta\omega}{\omega}$, resulting from a 1% increase in r.

c) A small element of volume in spherical polar coordinates is given by $dV = r\cos(\theta)d\varphi\ r\ d\theta\ dr$.

Evaluate $V = \int_0^{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^R r^2\cos(\theta)d\varphi\ d\theta\ dr$, thus calculating the volume of a sphere, radius R. **[20]**

8. **a)** Rewrite $z = 32\sqrt{2}(1+i)$ in exponential form. By considering that $f(\theta) = f(\theta + 2\pi) = f(\theta + 4\pi)$, find the three cube roots of z and sketch them on an Argand diagram.

b) The contibution to the moment of inertia of a circular lamina, of a small element at r, is given by: $dI = \rho r^3 dr d\theta$ where ρ is the mass per unit area, hence the moment of inertia is given by: $I = \rho\int_0^{2\pi}\int_0^R r^3 dr d\theta$, for a disk, radius R. Evaluate this integral and show that, for a disk of mass M = $\rho\pi R^2$ it is simply ½ MR^2. **[20]**

9. **a)** An ellipse, radii a,b, can be written as $x = a\cos\theta$ and $y = b\sin\theta$. Calculate the area of a quarter of the ellipse, between x = 0 (θ = π/2) and x = a (θ = 0).

b) An ellipse can also be described by: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

i) Evaluate $\dfrac{dy}{dx}$, by <u>implicit</u> differentiation.

ii) Calculate $\dfrac{dy}{dx}$ when $x = \dfrac{a}{\sqrt{2}}$ and $y = \dfrac{b}{\sqrt{2}}$.

c) Given that $I_n = \int\cos^n x dx$, prove that $nI_n = \sin x\cos^{n-1}x + (n-1)I_{n-2}$.

[20]

10. **a)** Evaluate the average value of $y = k\sin(2\pi x)$ in the interval $0 \le x \le 1$. Next evaluate its root mean square value. Comment on your results.

b) In a quantum mechanical system, comprising two energy levels with energy separation, ε, and at temperature, T, the relative occupancy of the upper level, $f(\varepsilon) = \frac{P_u}{P_l}$ is given by:

$$f(\varepsilon) = e^{\frac{-\varepsilon}{kT}}$$ where k is the Boltzmann constant.

If the energy gap increases slightly from ε to $\varepsilon + \delta\varepsilon$, isothermally (that is, the temperature is kept <u>constant</u>), then the new relative occupancy is given by $f(\varepsilon + \delta\varepsilon)$.

i) Write the first four terms of a Taylor series for $f(\varepsilon + \delta\varepsilon)$.

ii) Using the four terms of the Taylor expansion, and the expression for the relative change in occupancy.

$$\frac{\delta f}{f} = \frac{\delta f(\varepsilon)}{f(\varepsilon)} = \frac{f(\varepsilon + \delta\varepsilon) - f(\varepsilon)}{f(\varepsilon)}$$

Show that: $$\frac{\delta f}{f} = -\frac{\delta\varepsilon}{kT} + \frac{1}{2}\left(\frac{\delta\varepsilon}{kT}\right)^2 - \frac{1}{6}\left(\frac{\delta\varepsilon}{kT}\right)^3$$

[20]

End of Examination Paper

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