Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-121 Mathematical Techniques I

Time Allowed: 2 hours and 15 minutes

Date: **Thursday 03 May 2007**

Time: **10:00**

Answer all questions in section A and four questions from section B. An indicative marking scheme is shown in square brackets [ ] after each part of a question.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

NUMERIC CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

Useful information:

$$R=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},\qquad s=\int_{t=t_0}^{t=t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt,\qquad A=2\pi\int_{x=x_1}^{x=x_2}y\sqrt{1-\left(\frac{dy}{dx}\right)^2}\,dx,$$

$$\langle y\rangle=\frac{1}{b-a}\int_{x=a}^{x=b}y\,dx,\qquad \overline{x}=\frac{1}{A}\int_{x=x_1}^{x=x_2}xydx$$

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

Section A : Questions A1 to A4. Answer <u>all</u> questions.

A1) **i)** Differentiate the following with respect to x:

a) $(2x+1)^3$ **b)** $x^2 \sin x$ **c)** $e^{\cos(x)}$

d) $\ln(3x)$ **e)** $\dfrac{\alpha x}{(\beta x+\gamma)^2}$

ii) A curve is defined parametrically by $x = 2\theta + \theta^2$ and $y = 2\cos\theta$.

a) Calculate $\dfrac{dy}{d\theta}$.

b) Calculate $\dfrac{dx}{d\theta}$, and $\dfrac{d\theta}{dx}$.

c) Hence calculate $\dfrac{dy}{dx}$ in terms of θ.

d) Calculate the quantity $\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)$ in terms of θ.

e) Calculate the quantity $\dfrac{d^2y}{dx^2}$ in terms of θ.

[10]

A2) **i)** Write down all the terms in the series: $\sum_{n=1}^{n=4} \dfrac{x^n}{n}$.

ii) For a function, $f(x)$, write down the first four terms of its Maclaurin series expansion.

iii) For a function, $f(x)$, write down the first four terms of its Taylor series expansion about the point $x = a$.

iv) **a)** Evaluate $(2+i)-(3-2i)$.

b) Evaluate $(1-2i)(1+2i)$.

c) Evaluate $\dfrac{(-3-2i)}{(2-i)}$.

d) Express $(1+i)$ in the form $re^{i\theta}$.

e) For the complex number $z = 2e^{i\pi/3}$, calculate the roots $z^{1/3}$ and draw them on an Argand diagram.

[10]

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A3) **i)** Evaluate the following integrals

a) $\int x^3+3x^2+2x+1\,dx$,

b) $\int \frac{1}{x+1}\,dx$,

c) $\int \sin(2\pi x)dx$,

d) $\int_{x=0}^{x=1} 1+x\,dx$.

ii) Integrate $\int t\sin t\,dt$ by parts.

iii) Evaluate the following multiple integrals:

a) $\int_{x=0}^{1}\int_{y=0}^{1} (2x+1)y\;\,dydx$,

b) $\int_{y=0}^{b}\int_{x=0}^{a} y+\sin x\,dxdy$.

[10]

A4) **i)** Find all first and second partial derivatives of the function $z=3x^3+xy^2+3$.

ii) Given $I_n=\int x^n e^{-\gamma x}dx$, where γ is constant, prove that $I_n=-\frac{x^n e^{-\gamma x}}{\gamma}+\frac{n}{\gamma}I_{n-1}$.

Hence determine $\int x^2 e^{-3x}\,dx$.

[10]

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Section B: Questions B1 to B6. Answer <u>four</u> questions only.

B1) **i)** Find all stationary points of the function $y=\frac{x^3}{3}-x+5$. Sketch the function, and indicate any turning points and points of inflection.

ii) Calculate a radius of curvature of the function $y=x^3+3x-1$ at $x=2$.

iii) Calculate $\frac{dy}{dx}$ by implicit differentiation, for the ellipse defined by

$$R^2=\frac{x^2}{a}+by^2.$$

[15]

B2) **i)** For the complex number $z=1$,

a) Calculate z^3.

b) Calculate $z^{1/4}$, plotting all complex roots on an Argand diagram.

ii) A quantum mechanical process of an initial state decaying into a final state through two possible probability amplitudes $A_1=ae^{ib}$ and $A_2=ce^{id}$ has a total probability amplitude of $A_{\mathrm{TOTAL}}=A_1+A_2$.

a) Calculate the probability of the decay which is given by $P=A_{\mathrm{TOTAL}}A^*_{\mathrm{TOTAL}}$.

b) If the phase of the two amplitudes is the same, i.e. $b=d$, simplify your expression for the probability of the decay.

iii) The radius of a focussed beam of laser light of wavelength λ that passes through a lens of focal length f is given by

$$r_S=\frac{\lambda f}{\pi w_L},$$

where w_L is the beam radius at the lens. By suitable partial differentiation write down δr_s the change in r_s resulting from small changes in the wavelength and incident beam radius at the lens, $\delta\lambda$ and δw_L, respectively. Calculate the change $\delta r_s/r_s$ resulting from a change of 0.1% in both λ and w_L.

[15]

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B3) i) Show that the function $f(t) = A\sin\omega t$ satisfies a differential equation $f''(t) = Cf(t)$, and determine the value of the constant C.

ii) The velocity of a damped oscillator can be written as $v(t) = Ae^{-\lambda t}\cos\omega t$, where A, λ and ω are constants. The displacement of the oscillator as a function of time is given by

$$s(t) = \int v(t)\,dt.$$

Evaluate $s(t) = \int Ae^{-\lambda t}\cos\omega t\,dt$.

[15]

B4) i) The volume element of a sphere in spherical polar co-ordinates is given by $dV = r^2 \sin\theta d\theta d\phi dr$, where r is the distance of the volume element from the origin, θ lies between 0 and π, and $-\pi \le \phi \le \pi$.

a) By suitable integration, calculate the volume of a sphere of radius R.

b) The Earth has a radius of approximately 6,373 km. Assume that the crust of the Earth is on average 20 km thick, estimate the volume of the Earth's crust.

ii) The lamina $y = x^{1/2}$, between $x = 0$, and $x = 1$ generates a solid with a surface area A when rotated about the x-axis. By suitable integration, calculate A.

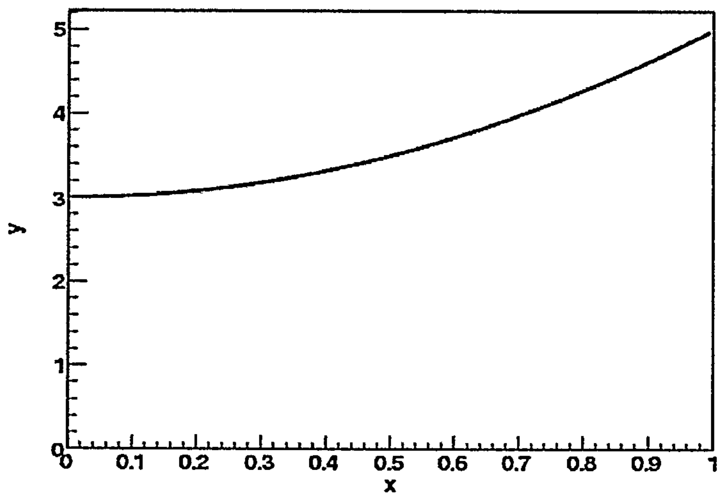

Figure 1: The lamina defined by $y = 2x^2 + 3$, between $x = 0$, and $x = 1$, and the x axis.

[15]

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B5) i) The moment of inertia about the y-axis of a rectangular strip of metal of height h, and length l (along the direction of the x-axis) is $dI = \rho h x^2 dx$, where ρ is the mass per unit area of the metal. By suitable integration, calculate the moment of inertia of the strip about the y-axis.

ii) Consider the lamina defined by $y = 2x^2 + 3$, between $x = 0$, and $x = 1$, and the x axis (see Figure 1).

a) Calculate the average height of the lamina (that is, the average value of y in the interval shown.)

b) Calculate the x co-ordinate of the centroid of the lamina.

iii) A solid is formed by the surface $z = xye^{-x}$, the x-y plane and the planes $x = 1$ and $y = 1$ (see Figure 2). Determine the volume of this solid in (unit)^3.

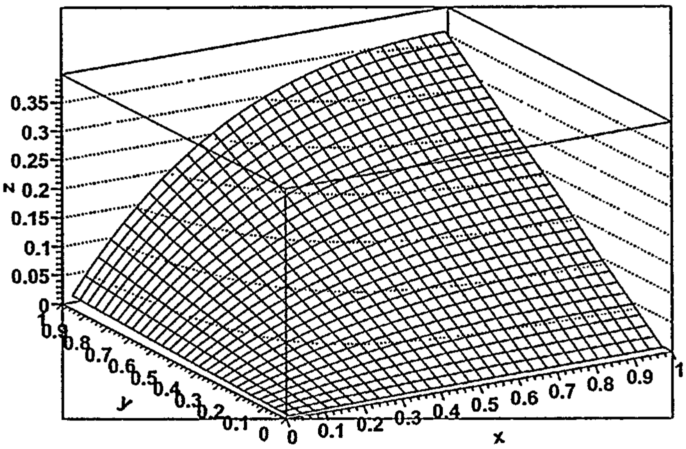

Figure 2: A solid formed by the surface $z = xye^{-x}$, the x-y plane and the planes $x = 0, x = 1,\ y = 0$ and $y = 1$. **[15]**

B6) The number of nuclei of a radioactive isotope in a sample at any given time t is

$$N(t) = N_0 e^{-\lambda t},$$

where N_0 is the number of radioactive nuclei at a time $t = 0$, and λ is the decay constant of the isotope. Both N_0 and λ are constants.

i) An experiment is started at a time $t = 0$ with N_0 radioactive nuclei. Write the first four terms of a Maclaurin series for $N(t)$.

ii) Using this result, estimate the fraction of nuclei that decay after a time $\ln 2/\lambda$ which is given by

$$\frac{N(t=0) - N(t=\ln 2/\lambda)}{N(t=0)}.$$

iii) The radioactive isotope ^{210}Po decays by the emission of α particles and has a decay constant $\lambda = 5.797\times10^{-8}$ s^{-1}. Using the series expansion obtained above, estimate how many ^{210}Po atoms remain after 40 days of running an experiment, if there are 10^6 atoms initially.

[15]

End of Examination Paper
Dr A. Bevan