Queen Mary
UNIVERSITY OF LONDON

M.Sc. Astrophysics

MAS402/ASTM112 Astrophysical Fluid Dynamics

Duration: 3 hours
10 May 2005 10:00

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best **3** *questions answered will be counted.*

Calculators ARE permitted in this examination.

Notation

The following notation is used throughout unless otherwise stated. The pressure, density, gravitational potential and adiabatic exponents are denoted by p, ρ, ψ, Γ_1 and Γ_3 respectively. The equilibrium values of these quantities are sometimes distinguished using a zero subscript. The position vector is denoted by $\mathbf{r}$ or $\mathbf{x}$, the time by t, the velocity by $\mathbf{u}$, the surface radius of a spherical configuration by R, and the gravitational constant by G. Vectors are denoted by boldface type.

Astronomical and Physical Data

Mass of the Sun	$M_\odot$	$2.0 \times 10^{30}\,\mathrm{kg}$
Surface radius of the Sun	$R_\odot$	$7.0 \times 10^{8}\,\mathrm{m}$
Luminosity of the Sun	$L_\odot$	$3.8 \times 10^{26}\,\mathrm{J\,s^{-1}}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{kg^{-1}m^{3}s^{-2}}$
Speed of light in a vacuum	c	$3.0 \times 10^{8}\,\mathrm{m\,s^{-1}}$

Standard Formulae

Candidates may assume the following set of basic equations and formulae:

In spherical polar coordinates (r, θ, ϕ)

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)$$

and

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\,.$$

For $\mathbf{u} = (u_r, u_\theta, u_\phi)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

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and

$$\nabla \times \mathbf{u} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta \mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ u_r & ru_\theta & r\sin\theta u_\phi \end{vmatrix} .$$

The spherical harmonic $Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\exp(im\phi)$, where $P_l^{|m|}$ denotes the associated Legendre function, satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_l^m}{\partial\phi^2} + l(l+1)Y_l^m = 0 ,$$

where l is a non-negative integer and m is an integer such that $|m| \leq l$. Further

$$\nabla^2(Y_l^m r^l) = 0 \qquad \nabla^2(Y_l^m r^{-l-1}) = 0 .$$

In cylindrical polar coordinates (r,ϕ,z), with $\mathbf{u} = (u_r, u_\phi, u_z)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}, \quad \nabla\times\mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ u_r & ru_\phi & u_z \end{vmatrix} .$$

The material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla .$$

The equation of motion for an inviscid fluid may be assumed in the form

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p - \rho\nabla\psi .$$

The continuity equation may be assumed in the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 .$$

The energy equation may be assumed in the form

$$\frac{Dp}{Dt} - \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt} = \rho(\Gamma_3 - 1)\left(\epsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}\right) ,$$

where ϵ is the heat generated per unit mass, and $\mathbf{F}$ is the heat flux. For adiabatic motion, the right-hand side of this equation is zero.

The gravitational potential satisfies Poisson's equation, $\nabla^2\psi = 4\pi G\rho$, which may be assumed to have the solution

$$\psi(\mathbf{r},t) = -\int_V \frac{G\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' ,$$

where the integration is taken over the fluid volume V, and $\mathrm{d}V'$ denotes the volume element $\mathrm{d}^3\mathbf{r}'$.

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1 Starting from the equation of motion for an inviscid fluid in an inertial frame of reference, derive the corresponding equation in a frame rotating steadily with angular velocity $\mathbf{\Omega}$. Hence, or otherwise, derive the equation

$$\frac{1}{\rho}\nabla p + \nabla\psi + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = 0$$

describing the dynamical equilibrium of a rotating fluid.

Show that $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$ can be expressed as a gradient of a scalar function **if and only if** the angular velocity is constant over cylinders centered about the axis of rotation ('rotation constant on cylinders').

Show that $(1/\rho)\nabla p$ can be expressed as a gradient of a scalar function **if and only if** $p(\mathbf{r})$ and $\rho(\mathbf{r})$ are such that pressure is a function of density only, i.e. $p = p(\rho)$ (a 'barotropic configuration').

Show further that $\mathbf{\Omega}$ is constant on cylinders **if and only if** the configuration is barotropic.

State, giving your reasoning, whether or not the condition of **thermal** equilibrium can be satisfied in a rotating star having $\mathbf{\Omega}$ constant on cylinders

(a) when the energy is transported by radiation (assume the radiative flux to be proportional to temperature gradient);

(b) the energy is transported by convection (assume the specific entropy to be uniform).

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2 The linear equations of adiabatic radial stellar oscillations (you can assume them here **without derivation**) are

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r}\right) U - \frac{1}{\rho_0 c^2} p_1, \tag{1}$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2 + 4\pi G\rho_0\right) \rho_0 U - \frac{g_0}{c^2} p_1, \tag{2}$$

where $U(r)$ and $p_1(r)$ describe radial displacements and Eulerian pressure perturbations, ω is angular frequency, $\rho_0(r)$ and $g_0(r)$ are equilibrium density and gravitational acceleration, c is adiabatic sound speed, and N is Brunt-Väisälä frequency:

$$c^2 = \Gamma_1 \frac{p_0}{\rho_0}, \quad N^2 = -g_0 \left(\frac{d \ln \rho_0}{dr} - \frac{1}{\Gamma_1} \frac{d \ln p_0}{dr}\right),$$

where $p_0(r)$ is equilibrium pressure and $\Gamma_1(r)$ is adiabatic exponent.

Using these equations, consider the radial oscillations of a star of uniform density, $\rho_0(r) = \text{const}$, and uniform adiabatic exponent, $\Gamma_1(r) = \text{const}$:

(a) Show that inside the star, the equation of hydrostatic equilibrium $dp_0/dr = -\rho_0 g_0$, combined with

$$g_0(r) = \frac{4}{3}\pi G \rho_0 r$$

and the requirement that $p_0(r)$ vanishes at the stellar surface $r = R$, leads to

$$p_0(r) = \frac{2}{3}\pi G \rho_0^2 \left(R^2 - r^2\right).$$

(b) Show further that

$$c^2(r) = \frac{2}{3}\pi G \rho_0 \Gamma_1 \left(R^2 - r^2\right), \quad \text{and} \quad N^2(r) = -\frac{g_0^2(r)}{c^2(r)}.$$

Now show by direct substitution into the oscillation equations (1,2) that

$$U(r) = Ar,$$

where A is an arbitrary normalization constant, is an eigenfunction, for suitable eigenfrequency ω which you should determine using the following steps:

Using equation (1), show that $U(r) = Ar$ gives

$$p_1(r) = A\rho_0 \left(g_0 r - 3c^2\right).$$

Using equation (2), show that your $U(r)$ and $p_1(r)$ solve the oscillation equations (1,2) if, and only if,

$$\omega^2 = \left(\Gamma_1 - \frac{4}{3}\right) 4\pi G \rho_0 = (3\Gamma_1 - 4) \frac{GM}{R^3},$$

where M is stellar mass. Discuss briefly the physical nature of this solution.

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(a) Propagation of acoustic waves in gas with uniform density, pressure and sound speed is governed by the wave equation (which you can assume without derivation)

$$\frac{\partial^2 p'}{\partial t^2} = c^2 \nabla^2 p'.$$

With x as one of the cartesian coordinates, show that $p' = f(ct - x)$ and $p' = f(ct + x)$ satisfy the wave equation for any analytic function $f(z)$.

A spherically symmetric acoustic wave is emitted by a small pulsating source. Show that the wave amplitude decreases with distance r from the source as $1/r$. [Hint: try $p' = Ar^n \exp(ikr - i\omega t)$.]

(b) Incompressible fluid in uniform gravitational field with gravitational acceleration g directed downwards (along z-axis) has density ρ_1 for $z < 0$ and ρ_2 for $z > 0$. Neglecting gravity perturbations, show that the dispersion relation for waves propagating in a horizontal direction along the interface $z = 0$ is

$$\omega^2 = kg\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1},$$

where k is horizontal wavenumber. What is the physical meaning of the solution when $\rho_1 > \rho_2$?

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4 A fictitious star of mass M and radius R is composed of a homogeneous (ρ_0 constant) incompressible ($c^2 = \infty,\ \nabla \cdot \mathbf{u} = 0$) fluid. The star undergoes linear, non-radial oscillations.

(a) Write down the linear perturbation equations relevant to the oscillations of such a homogeneous, incompressible star. Show that the Eulerian pressure perturbation p' and perturbation to the gravitational potential ψ' both satisfy Laplace's equation. The outer mechanical boundary condition is that the Lagrangian pressure perturbation should vanish. Derive from this a relation between p' and the radial component u_r of the velocity at the surface $r = R$.

(b) By considering the pressure perturbation p', or otherwise, show that in the Cowling approximation (i.e. neglecting ψ') the oscillation frequencies of modes of degree ℓ are given by

$$\omega^2 = \ell \frac{GM}{R^3}.$$

(c) Also at the stellar surface, the gravitational potential ψ and its gradient $\nabla\psi$ must be continuous. Show that these conditions imply that

$$[\psi']_-^+ = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\left[\frac{\partial \psi'}{\partial r}\right]_-^+ = 4\pi G \rho_0 u_r,$$

where for any function $f(r)$ we denote $[f]_-^+ = \lim_{\epsilon \to 0}[f(R+\epsilon) - f(R-\epsilon)]$. Hence show that when the perturbations of the gravitational potential are taken into account, the oscillation frequencies of the modes of degree ℓ are

$$\omega^2 = \frac{2\ell(\ell-1)}{2\ell+1}\frac{GM}{R^3}.$$

(d) Discuss briefly your results and their possible applicability. The complete analysis (part c) predicts $\omega = 0$ for dipole ($\ell = 1$) modes. How do you interpret such a peculiar result?

5 Write briefly on **two** of the following topics:

(a) rotation and mixing in a solar-type star;

(b) solar and stellar seismology;

(c) the effects of rotation and magnetic fields on star formation;

(d) close binary stars.

[End of paper]
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