Queen Mary
UNIVERSITY OF LONDON

B.Sc. EXAMINATION BY COURSE UNITS

MAS322 Relativity

14 May 2007, 10am
Time Allowed: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

The question paper must not be removed by candidates from the examination room. You must not start reading the question paper until instructed to do so.

You are reminded of the following information, which you may use without proof.

The metric tensor of special relativity is η_{ab} such that

$$ds^2 = \eta_{ab}dx^a dx^b = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

The Lorentz transformations between two frames F and F' in standard configuration are

$$x' = \gamma(x - Vt), \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right), \quad y' = y, \quad z' = z$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$ and F' is moving with speed V relative to F.

Partial derivatives:

$$Q_{,a} = \frac{\partial Q}{\partial x^a}$$

Covariant derivatives are denoted by semicolon: e.g. $V_{a;b} = V_{a,b} - \Gamma^c{}_{ab}V_c$

The Riemann curvature tensor:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}$$

Euler–Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

Geodesic equation:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

SECTION A: *You should attempt all questions. Marks awarded are shown next to the questions.*

1. A person drives a car 4 metres in length at a speed $0.8c$. Calculate the length of the car as measured by an observer who is stationary with respect to the car.

Suppose the rest-mass of the car is 1 tonne, but that the stationary observer measures its mass to be 5 tonnes. How fast is the car travelling in this case?

[You may assume both observers are in a Standard Configuration]. [8]

2. Consider the 4-vector, $\bar{V}$, with components

$$\bar{V} = \sqrt{2}\bar{e}_0 + \sqrt{7}\bar{e}_1$$

where the basis vectors $\bar{e}_0$ and $\bar{e}_1$ are given by $\bar{e}_0 = (1,0,0,0)$ and $\bar{e}_1 = (0,1,0,0)$.

Is $\bar{V}$ a spacelike or timelike 4-vector?

Determine appropriate values of the components W^0 and W^1 of the 4-vector $\bar{W} = (W^0, W^1, 0, 0)$ such that $\bar{V}$ and $\bar{W}$ are orthogonal.

Consider a 4-vector of the form $\bar{Y} = (Y^0, Y^1, Y^2, Y^3)$. Derive a condition on the component Y^0 in terms of the components (Y^1, Y^2, Y^3) for $\bar{Y}$ to be a unit, timelike vector. [10]

3. Show, taking care to explain your reasoning, that $X^{ab}Y_{ab} + g^{ab}g_{ab} = 4$, where X^{ab} is an arbitrary symmetric tensor, Y_{ab} is an arbitrary antisymmetric tensor and g_{ab} is the metric tensor. [12]

4. Write down the transformation law under a general coordinate transformation for a type (1,2) tensor, $M^a{}_{bc}$.

Show that the contraction of a type $(1,1)$ tensor, $R^a{}_b$, results in a scalar, i.e., a type $(0,0)$ tensor. [10]

5. Consider a Local Inertial Frame at a point P in spacetime. What are the values of the components of the metric tensor, g_{ab}, and the connection, $\Gamma^a{}_{bc}$, at the point P?

[This question continues overleaf . . .]

Using the expression for the Riemann curvature tensor given on page (1), show that in a Local Inertial Frame

$$R^a{}_{bcd} = -R^a{}_{bdc}$$

Explain why this relation is valid in an arbitrary frame of reference. [10]

SECTION B: *Each question carries 25 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.*

6. (a) A particle of rest mass m_1 moves along the x-axis with a speed u_1 and collides with a particle of rest mass m_2 moving in the same direction with speed u_2. After the collision, both particles continue to move in the same direction so that particle m_1 now has a speed u_1' and particle m_2 has a speed u_2'.

Prove that

$$\frac{\gamma(u_1)\gamma(u_2)}{\gamma(u_1')\gamma(u_2')} = \frac{1 - u_1'u_2'}{1 - u_1 u_2}$$

where $\gamma(u) \equiv (1 - u^2)^{-1/2}$.

(b) A particle of rest mass M moving along the x-axis with speed v decays into two particles each with a rest mass $M/2$. Both particles continue to move along the x-axis.

Show that the new particles move with the same speed. Show also that the speed of the new particles equals that of the original particle.

[In this question you may set the speed of light $c = 1$].

7. The metric for a particular two–dimensional Riemannian spacetime is given by

$$ds^2 = -y^2 dx^2 + x^2 dy^2$$

Employ the geodesic equation to calculate all the components of the connection $\Gamma^a{}_{bc}$ for this metric. Hence calculate the $R^x{}_{yxy}$ component of the Riemann tensor.

[This question continues overleaf . . .]

8. Consider the scalar invariant W^aV_a, where W^a is an arbitrary contravariant vector and V_a is an arbitrary covariant vector. Given the form of the covariant derivative for V_a on page (1), derive the form of the covariant derivative for W^a.

If S_{ab} is an arbitrary antisymmetric tensor, write down the covariant derivative $S_{ab;c}$ in terms of the partial derivative of S_{ab} and the connection. Hence show that

$$S_{ab;c} + S_{bc;a} + S_{ca;b} = S_{ab,c} + S_{bc,a} + S_{ca,b}$$

The Einstein Field Equations are written in the form

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}$$

where κ is a constant.

Give the definitions of the quantities R_{ab} and R and explain the physical significance of the tensor T_{ab}.

Show by using the field equations that if $T_{ab} = \alpha R_{ab}$, where α is an arbitrary constant such that $|\alpha\kappa| \neq 1$, it follows that $R = 0$ and $R_{ab} = 0$.

If $R_{ab} = \beta g_{ab}$, where β is an arbitrary constant, find a relation between T_{ab} and g_{ab}.

9. The Schwarzschild spacetime metric given in standard form is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

where G and M are constants.

Write down the conditions for a test particle to follow a timelike, radial trajectory in this spacetime. What are the conditions for the trajectory to be circular?

The components of the connection are given by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left[g_{db,c} + g_{dc,b} - g_{bc,d}\right]$$

Employ this formula to calculate the connection components $\Gamma^t{}_{rt}$ and $\Gamma^t{}_{rr}$ for the Schwarzschild metric.

Show, by deriving the appropriate Euler-Lagrange equation, that a particle following a path in the plane $\theta = \pi/2$ remains in that plane.

Determine the value of r in terms of G and M such that a particle moving in a circular orbit with $\theta = \pi/2$ completes one orbit in a time $t = 2\pi GM$.

[End of examination paper]