MSci 4261 ELECTROMAGNETISM

Week 9: Answers

1. In the cited section 9.3 of the lecture notes there is a formula for the electric field associated with an accelerated particle. The first term (the velocity field) may be dropped, since we are interested only in the field far from the particle, and the acceleration field dominates there. Furthermore, we are told that the acceleration is parallel to the velocity, so that $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$, and what is left is what was asked to be shown.

2. The magnetic field may be derived from $\mathbf{B} = [\mathbf{n} \times \mathbf{E}]_{\mathrm{ret}}/c$, and then we have

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \\ &= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c}\right)^2 \left[\frac{\mathbf{n} \times [\mathbf{n} \times \dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{\mathrm{ret}} \times \left\{\mathbf{n} \times \left[\frac{\mathbf{n} \times [\mathbf{n} \times \dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{\mathrm{ret}}\right\}\end{aligned}$$

and after use of the identity

$$\left(\mathbf{n} \times [\mathbf{n} \times \dot{\boldsymbol{\beta}}]\right) \times \left(\mathbf{n} \times \left(\mathbf{n} \times [\mathbf{n} \times \dot{\boldsymbol{\beta}}]\right)\right) = [\mathbf{n} \times \dot{\boldsymbol{\beta}}]^2 \mathbf{n}$$

this gives the desired result.

3. We need to use

$$\frac{dt}{dt'} = 1 - \boldsymbol{\beta}\cdot\mathbf{n} = 1 - \beta\cos\theta,$$

and the result follows.

4. In the non-relativistic limit, the above becomes

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\dot{u}^2\sin^2\theta,$$

as required.

5. The angular dependence is in the factors

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5},$$

and so the maximum intensity occurs in the direction where this has a maximum, given by simple calculus as where

$$2\sin\theta\cos\theta(1-\beta\cos\theta) = 5\sin^2\theta\,\beta\sin\theta,$$

which gives a quadratic equation for $\cos\theta$;

$$3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0.$$

There is only one root with a real value for θ, and that is the one given. Since $\beta = \sqrt{(1-\gamma^{-2})}$, we have

$$\begin{aligned}
\cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right] &\approx \cos^{-1}\left[\frac{1}{3}(1+\frac{1}{2\gamma^2})\left(4(1-\frac{15}{32\gamma^2})-1\right)\right] \\
&= \cos^{-1}\left[(1+\frac{1}{2\gamma^2})\,(1-\frac{5}{8\gamma^2})\right] \\
&\approx \cos^{-1}\left[1-\frac{1}{8\gamma^2}\right] \\
&\approx \frac{1}{2\gamma}
\end{aligned}$$

which was to be shown.

The polar plots are shown in Jackson, *Classical Electrodynamics*, Fig. 14.4.