MSci 4261 ELECTROMAGNETISM EXERCISES
Answers: Week 8

1. Using the expressions given, one finds directly:

$$\begin{aligned}
\mathbf{E}'\cdot\mathbf{B}' &= E'_xB'_x + E'_yB'_y + E'_zB'_z \\
&= E_xB_x + (\cosh\zeta\ E_y - \sinh\zeta\ cB_z)(\cosh\zeta\ B_y + \sinh\zeta\ E_z/c) \\
&\qquad + (\cosh\zeta\ E_z + \sinh\zeta\ cB_y)(\cosh\zeta\ B_z - \sinh\zeta\ E_y/c) \\
&= E_xB_x + E_yB_y + E_zB_z \\
&= \mathbf{E}\cdot\mathbf{B},
\end{aligned}$$

with the cross-terms cancelling and the others combining using $\cosh^2 - \sinh^2 = 1$. Similarly:

$$\begin{aligned}
\mathbf{E}'^2 - c^2\mathbf{B}'^2 &= E_x^2 + (\cosh\zeta\ E_y - \sinh\zeta\ cB_z)^2 + (\cosh\zeta\ E_z + \sinh\zeta\ cB_y)^2 \\
&\quad - c^2B_x^2 - c^2(\cosh\zeta\ B_y + \sinh\zeta\ E_z/c)^2 - c^2(\cosh\zeta\ B_z - \sinh\zeta\ E_y/c)^2 \\
&= E_x^2 + E_y^2 + E_z^2 - c^2B_x^2 - c^2B_y^2 - c^2B_z^2 \\
&= \mathbf{E}^2 - c^2\mathbf{B}^2.
\end{aligned}$$

2.

(a) Writing I_1 for $\mathbf{E}\cdot\mathbf{B}$ and I_2 for $\mathbf{E}^2 - c^2\mathbf{B}^2$, one finds that if $\mathbf{E}$ and $\mathbf{B}$ are parallel, $c^2B^4 + I_2B^2 - I_1^2 = 0$. Since $I_2^2 + 4I_1^2c^2$ is clearly never negative, this gives two real solutions for B^2, and unless $I_1 = 0$, one and only one of these is positive, namely $B^2 = (\sqrt{I_2^2 + 4I_1^2c^2} - I_2)/2c^2$. The existence of this solution confirms that there is no obstacle to finding a frame in which $\mathbf{E}$ and $\mathbf{B}$ are parallel, unless $I_1 = 0$ in which case $\mathbf{E}$ and $\mathbf{B}$ would be perpendicular in *every* frame; and in that case the only way that they could also be parallel would be if one or the other vanishes— and that situation is dealt with below.

(b) For there to exist a frame in which $\mathbf{B} = 0$ it is clear that we require $I_1 = 0$ and $I_2 \geq 0$. If in fact $I_2 = 0$ it would follow that $\mathbf{E}$ also vanishes, and we would have $\mathbf{E} = \mathbf{B} = 0$ in every frame.

(c) Likewise for there to exist a frame in which $\mathbf{E} = 0$, excluding the case when $\mathbf{E} = \mathbf{B} = 0$ in every frame it is necessary and sufficient that $I_1 = 0$ and $I_2 < 0$.

(d) In this case $I_1 = 0 = I_2$ in every frame but $\mathbf{B}$ and $\mathbf{E}$ are non-zero in every frame.

3. We know that

$$P^\mu = (E/c, \mathbf{p})$$

is a 4-vector and that $d\tau$ is a scalar, where

$$c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

so that

$$\frac{d\tau}{dt} = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{\frac{1}{2}} = \frac{1}{\gamma}.$$

It follows that $\frac{dP^\mu}{d\tau}$ is a 4-vector, *i.e.*

$$\begin{aligned}
\frac{dP^\mu}{dt}\frac{dt}{d\tau} &= \gamma\frac{dP^\mu}{dt} \\
&= \gamma\left(\frac{dP^0}{dt}\ ,\ \mathbf{f}\right),
\end{aligned}$$

and since $cP^0 = E$ and its time-derivative is just the rate at which the force $\mathbf{f}$ does work, *viz.* $c\frac{dP^0}{dt} = \mathbf{v}\cdot\mathbf{f}$, we have obtained the desired result. We have also identified F^0 as the proper time derivative of P^0, *i.e.* up to a factor of c of the energy.