MSci 4261 ELECTROMAGNETISM EXERCISES

Week 8

Reminder: The Lorentz transformation from a frame K to a frame K' moving with constant velocity $\mathbf{v}$ along the x-axis relative to K is given by:

$$\begin{aligned} x'^0 &= \cosh\zeta \;\; x^0 - \sinh\zeta \;\; x^1, \\ x'^1 &= -\sinh\zeta \;\; x^0 + \cosh\zeta \;\; x^1, \\ x'^2 &= x^2, \\ x'^3 &= x^3, \end{aligned}$$

where ζ is the *rapidity* defined by $\tanh\zeta = \frac{v}{c}$.
The corresponding transformations for the components of the electric and magnetic fields are given by:

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \cosh\zeta \;\; E_y - \sinh\zeta \;\; cB_z \\ E'_z &= \cosh\zeta \;\; E_z + \sinh\zeta \;\; cB_y, \end{aligned}$$

and

$$\begin{aligned} B'_x &= B_x \\ B'_y &= \cosh\zeta \;\; B_y + \sinh\zeta \;\; E_z/c \\ B'_z &= \cosh\zeta \;\; B_z - \sinh\zeta \;\; E_y/c. \end{aligned}$$

1. Use these equations to confirm directly what was shown in lectures using the transformation laws for the field tensor $F^{\mu\nu}$ and its dual ${}^\star F_{\alpha\beta}$, namely that $\mathbf{E}\cdot\mathbf{B}$ and $\mathbf{E}^2 - c^2\mathbf{B}^2$ are Lorentz invariants.

2. Given that these are the only independent invariants that can be constructed from $\mathbf{E}$ and $\mathbf{B}$ explain in what circumstances it is possible to find reference frames in which locally:

(a) $\mathbf{E}$ is parallel to $\mathbf{B}$,

(b) $\mathbf{B} = 0$,

(c) $\mathbf{E} = 0$,

(d) $\mathbf{E}$ and $c\mathbf{B}$ are equal in magnitude and perpendicular in all frames.

3. If $\mathbf{f} = \frac{d}{dt}\mathbf{p}$ is the force 3-vector where $\mathbf{p} = m\gamma\mathbf{v}$ and $\mathbf{v}$ are respectively the momentum and velocity 3-vectors, show that

$$F^\mu = \gamma(c^{-1}\mathbf{v}\cdot\mathbf{f}\;,\;\mathbf{f})$$

is a 4-vector. As usual $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.
How would you interpret F^0?