MSci 4261 ELECTROMAGNETISM EXERCISES

Week 4: Answers

1. Let $y = cx$, then by definition $1 = \int_{-\infty}^{\infty} dy\, \delta(y) = \int_{-\infty}^{\infty} dx\, c\, \delta(cx)$. Comparing this with the property that $1 = \int_{-\infty}^{\infty} dx\, \delta(x)$ gives the result.

2. Without choosing a gauge fixing condition (such as the Lorentz gauge), the equation for the scalar potential (in the absence of sources) yields $a_0 k^2 = \mathbf{k} \cdot \mathbf{a}\omega$. Substituting this into the equation for the vector potential then gives $(k^2 - \omega^2/c^2)(\mathbf{a} - \frac{\mathbf{k}\cdot\mathbf{a}}{k^2}\mathbf{k}) = 0$. Vanishing of the second factor implies that the fields are *pure gauge*, ie that the electric and magnetic fields are zero, as is easily checked. Thus the first factor must vanish, ie we have that $k^2 = \omega^2/c^2$. (In a Lorentz gauge, the equations satisfied by the potentials in the absence of sources are

$$\Box \mathbf{A} \equiv \left[\frac{\partial^2}{c^2 \partial t^2} - \nabla^2\right]\mathbf{A} = 0, \qquad \Box \Phi = 0.$$

Thus with the potentials as given, since

$$\Box \cos(\omega t - \mathbf{k} \cdot \mathbf{x}) = -(\omega^2/c^2 - \mathbf{k}^2)\cos(\omega t - \mathbf{k} \cdot \mathbf{x}),$$

we require $\omega^2 = c^2 k^2$.)

3. The Lorentz gauge condition

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

implies that $\mathbf{a} \cdot \mathbf{k} - \frac{1}{c} a^0 k = 0$. The radiation gauge condition is likewise satisfied if $\mathbf{k} \cdot \mathbf{a} = 0$, or $\boldsymbol{\epsilon} \cdot \mathbf{k} = 0$. If this is also imposed, it requires that $a^0 = 0$.

4. We thus have $\Phi = 0$, zero scalar potential, and $\mathbf{A} = \mathbf{a}\ \cos[\omega t - \mathbf{k} \cdot \mathbf{x}]$ for the vector potential. This gives

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} = \omega \mathbf{a}\ \sin[\omega t - \mathbf{k} \cdot \mathbf{x}],$$

which indeed oscillates with angular frequency ω and amplitude $a\omega$. Likewise

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{k} \times \mathbf{a}\ \sin[\omega t - \mathbf{k} \cdot \mathbf{x}],$$

giving an (in phase) oscillation with amplitude $|\mathbf{k} \times \mathbf{a}|$; since $\mathbf{k}$ and $\mathbf{a}$ are perpendicular this is just ak which in turn equals $a\omega/c$ because of the condition first found.

5. These electromagnetic fields oscillate in space and in time through their proportionality to $\sin[\omega t - \mathbf{k} \cdot \mathbf{x}]$, and so may be regarded as waves. Their surfaces of constant phase are given by

$$\mathbf{k} \cdot \mathbf{x} = \omega t + \text{constant}.$$

At any instant in time, this is the equation of a plane perpendicular to $\mathbf{k}$, and as time progresses, these planes advance in the direction of $\mathbf{k}$ with speed $\omega/k = c$.