MSci 4261 ELECTROMAGNETISM EXERCISES

Week 1: Answers

1. The first part is a straightforward application of vector calculus using Maxwell's equations. The significance of the term on the left-hand side of the equation

$$\int_S \mathcal{P}\cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \mathcal{E}.$$

is that it represents the flow of energy per unit time, out through the surface S, whilst the term on the right-hand side represents the corresponding change in the energy inside the volume V. In other words, this is the expression of conservation of energy.

2. Consider Maxwell's equations in vacuum (take $\epsilon_0 = 1$ and $\mu_0 = 1$ for simplicity)

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t},$$
$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}.$$

Suppose we take the matrix M to be equal to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then preservation of the Maxwell equations requires that $a = d, b = -c$, then preservation of the energy density and Poynting vector requires that $a^2 + b^2 = 1$. Whence, parameterising a by $a = \cos\theta$, M is a 2×2 rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

3. (See Jackson, Sec. 6.11) Maxwell's equations in the presence of electric and magnetic sources would be:

$$\nabla\cdot\mathbf{E} = \rho_e, \quad \nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{J_e},$$
$$\nabla\cdot\mathbf{B} = \rho_m, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - \mathbf{J}_m.$$

These too have rotational symmetries of exactly the type in Question 2. (Note that these symmetries mean that only the *ratio* of electric to magnetic charge is a meaningful concept in this context.) The new conservation law is conservation of magnetic charge.