MSci 4261 ELECTROMAGNETISM EXERCISES

Week 1

1. Consider the free space definitions of the energy density and Poynting vector:

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2,$$
$$\mathcal{P} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).$$

Show that Maxwell's equations imply that these satisfy the equation

$$\boldsymbol{\nabla} \cdot \mathcal{P} + \frac{\partial \mathcal{E}}{\partial t} = 0.$$

Show that this equation implies that (V is a region of space bounded by the surface S)

$$\int_S \mathcal{P} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \mathcal{E}.$$

Explain the physical significance of the terms in this equation.

2. Consider Maxwell's equations in a vacuum (take $\epsilon_0 = 1$ and $\mu_0 = 1$ for simplicity)

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t},$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Suppose that we introduce a 2×2 matrix M, and define new fields by

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = M \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}.$$

For which matrices M do the new fields still satisfy the Maxwell equations above, and also preserve the form of the energy density and Poynting vector?

3. Consider Maxwell's equations in the presence of electric sources (take $\epsilon_0 = 1$ and $\mu_0 = 1$ for simplicity):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \quad \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J},$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t}.$$

Suppose that magnetic monopoles exist. How would you modify these equations to consistently take account of magnetic sources? Do these new equations possess symmetries of the type in Question 2 above? Which new conservation equations are consequences of the new equations?