MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date: 22nd May 2006

Time: 10:00

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

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1. In the dipole approximation for a scattering centre at the origin, the electric and magnetic fields for the scattered radiation are given by

$$\mathbf{E}_{\rm SC} = \tfrac{k^2}{4\pi\epsilon_0}\tfrac{e^{ikr}}{r}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c],$$

$$\mathbf{B}_{\rm SC} = \mathbf{n}\times\mathbf{E}_{\rm SC}/c;$$

where $\mathbf{p}$ and $\mathbf{m}$ are the induced electric dipole and magnetic dipole moments of the scatterer. If the incident wave is a plane wave given by

$$\mathbf{E}_{\rm in} = \mathbf{E}_0 e^{i\mathbf{k}_0\cdot\mathbf{x}},$$

$$\mathbf{B}_{\rm in} = \mathbf{n}_0\times\mathbf{E}_{\rm in}/c,$$

with $\mathbf{k}_0 = k\mathbf{n}_0$, the differential scattering cross-section may be written as

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \frac{r^2\langle|\mathbf{S}_{\rm SC}|\rangle}{\langle|\mathbf{S}_{\rm in}|\rangle},$$

where $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ is the Poynting flux vector and the notation $\langle\cdots\rangle$ indicates time-averaging.

(a) Show that this reduces to

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2\frac{1}{E_0^2}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c]^2. \quad \text{[5 marks]}$$

(b) Now consider a collection of identical dipole scattering centres, located at the points $\mathbf{x}_j$. Show that the effect is to multiply the cross-section for a single scatterer by the structure factor

$$\mathcal{F}(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2, \quad \text{[5 marks]}$$

where $\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n})$.

(c) Show that for N scatterers $\mathcal{F}(0) = N^2$, and find an approximation for $\mathcal{F}(\mathbf{q})$ for $N >> 1$ scatterers distributed at random, with a a typical distance apart, for $|\mathbf{q}|a >> 1$. [5 marks]

(d) Explain what happens if the scatterers are spaced regularly, as for example in a crystal. [5 marks]

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2. (a) Consider spherical coordinates (r, θ, ϕ), and a vector field $\mathbf{A}$ with components

$$A_r = 0, \quad A_\theta = 0, \quad A_\phi = \frac{g(1-\cos\theta)}{4\pi r \sin\theta}.$$

Show that this potential gives a magnetic field

$$\mathbf{B} = \frac{g}{4\pi r^2}\hat{\mathbf{r}}.$$

[5 marks]

(b) Explain what this magnetic field describes. [3 marks]

(c) The Lorentz force law for the motion in this field of a particle of electric charge e, rest mass m, velocity $\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ and momentum $\mathbf{p} = \gamma(v)m\mathbf{v}$ gives

$$\dot{\mathbf{p}} = \frac{eg}{4\pi}\frac{\mathbf{v}\times\mathbf{r}}{r^3}.$$

Show that the quantities

$$E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}, \qquad \mathbf{J} = \gamma(v)m\mathbf{r}\times\mathbf{v} - \frac{eg}{4\pi}\frac{\mathbf{r}}{r},$$

are constants of the motion and explain what these invariants are physically and what the separate terms in $\mathbf{J}$ represent. [8 marks]

(d) Consider the case where the particle in part (c) above is stationary. Assuming that $\mathbf{J}$ has the properties of intrinsic angular momentum, derive the quantisation condition

$$\frac{eg}{4\pi} = \frac{n}{2}\hbar, \qquad n = 0, \pm1, \pm2, ...$$

[4 marks]

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3. The vector potential $\mathbf{A}(\mathbf{x})e^{-i\omega t}$ far from an oscillating magnetic dipole $\mathbf{m}e^{-i\omega t}$ at the origin is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r} \mathbf{n} \times \mathbf{m}.$$

(a) Define k, r and $\mathbf{n}$ in this equation. [3 marks]

(b) What is the magnetic field $\mathbf{B}$ at a distance which is far from the oscillating dipole? [5 marks]

(c) The Poynting vector $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{c}{\mu_0} |\mathbf{B}|^2 \mathbf{n}.$$

Show that this reduces to

$$\mathbf{S} = \frac{\mu_0}{16\pi^2} \frac{\omega^4}{c^3} \frac{1}{r^2} |\mathbf{m}|^2 \sin^2\theta \, \mathbf{n}.$$

What is the angle θ in this expression? [5 marks]

(d) A neutron star rotates with angular rotation frequency ω. It has a magnetic dipole moment of magnitude m, but this is misaligned with the axis of rotation by a constant angle α. Show that it radiates energy at a rate

$$\frac{dE}{dt} = -\frac{\mu_0}{6\pi} \frac{\omega^4}{c^3} m^2 \sin^2\alpha.$$

[7 marks]

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4. The Liénard-Wiechert potentials for the electromagnetic fields generated by a charge q following a trajectory $\mathbf{r} = \mathbf{r}(t)$, with instantaneous velocity $\mathbf{u} = \frac{d\mathbf{r}}{dt} = c\boldsymbol{\beta}$, are

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}},$$
$$\mathbf{A} = \frac{\mu_0 qc}{4\pi}\left[\frac{\boldsymbol{\beta}}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}}.$$

(a) Explain the meaning of the notation $[\ldots]_{\text{ret}}$, and define the distance R and the direction vector $\mathbf{n}$. [4 marks]

(b) If $|\boldsymbol{\beta}| << 1$, show that at large distances from the charge the electric field is

$$\mathbf{E}_{\mathbf{far}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{1}{R}(\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}))\right]_{\text{ret}}.$$ [6 marks]

(c) Assuming that the corresponding magnetic field is given by

$$\mathbf{B}_{\mathbf{far}} = [\mathbf{n}\times\mathbf{E}_{\mathbf{far}}]_{\mathbf{far}}/c,$$

show that at large distances, the Poynting energy-flux vector is

$$\mathbf{S}_{\mathbf{far}} = \frac{1}{\mu_o c}|\mathbf{E}_{\mathbf{far}}|^2\mathbf{n}.$$ [4 marks]

(d) Derive the Larmor formula

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}|\dot{\mathbf{u}}|^2$$

for the total instantaneous power radiated by a non-relativistic accelerated charge. [6 marks]

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5. (a) [3 marks] Show in the Lorentz gauge ($\partial^\mu A_\mu = 0$), with $A^\mu = (\frac{1}{c}\Phi, \mathbf{A})$ and $j^\mu = (c\rho, \mathbf{J})$, that the Maxwell equation $\partial^\mu F_{\mu\nu} = \mu_0 j_\nu$ reduces to

$$\partial^\mu \partial_\mu \mathbf{A} = \mu_0 \mathbf{J}, \qquad \partial^\mu \partial_\mu \Phi = \frac{\mathbf{1}}{\epsilon_0}\rho.$$

(b) [4 marks] Integrate the equation for $\mathbf{A}$ above with $\int_{-\infty}^{\infty} e^{-i\omega t}$ to obtain the Fourier transformed equation

$$(\nabla^2 + k^2)\mathbf{A}(\mathbf{x}, \omega) = -\mu_0 \mathbf{J}(\mathbf{x}, \omega), \qquad (1)$$

with $k^2 = \omega^2/c^2$.

(c) [3 marks] Suppose that there exists a Green function $G_k(\mathbf{x}, \mathbf{x}')$, satisfying

$$(\nabla^2 + k^2)G_k(\mathbf{x}, \mathbf{x}') = -4\pi\delta^3(\mathbf{x} - \mathbf{x}'). \quad (2)$$

Show that

$$\mathbf{A}(\mathbf{x}, \omega) = \frac{\mu_0}{4\pi} \int \mathbf{G_k}(\mathbf{x}, \mathbf{x}')\mathbf{J}(\mathbf{x}', \omega)\mathbf{d^3x'}$$

solves equation (1) above.

(d) [5 marks] Give an argument why $G_k(\mathbf{x}, \mathbf{x}')$ must be purely a function of $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$. Show that in this case equation (2) becomes

$$\frac{1}{r}\frac{d^2}{dr^2}(rG_k(r)) + k^2 G_k(r) = -4\pi\delta^3(\mathbf{r})$$

and hence that when $r \neq 0$, $G_k(r)$ is given by

$$G_k(r) = \frac{1}{r}(Ae^{ikr} + Be^{-ikr}), \quad (3)$$

for some constants A, B.

(e) [5 marks] A solution of Poisson's equation $\nabla^2\phi = -\frac{1}{\epsilon_0}\rho$ is $\phi = \frac{1}{4\pi\epsilon}\int \frac{\rho(\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$. Use this fact to show that when $r \to 0$, (3) above remains a solution of equation (2) if

$$A + B = 1.$$

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Formula Sheet

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c},\\
\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a},\\
\nabla\times(\psi\mathbf{a}) &= (\nabla\psi)\times\mathbf{a}+\psi(\nabla\times\mathbf{a}),\\
\nabla\times(\nabla\times\mathbf{a}) &= \nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a},\\
\nabla\big(\psi(r)\big) &= \mathbf{n}\psi'(r).
\end{aligned}$$

Maxwell's equations:

$$\begin{aligned}
\nabla\cdot\mathbf{B}=0, &\qquad \nabla\times\mathbf{E}=-\tfrac{\partial\mathbf{B}}{\partial t};\\
\nabla\cdot\mathbf{D}=\rho, &\qquad \nabla\times\mathbf{H}=\mathbf{j}+\tfrac{\partial\mathbf{D}}{\partial t}.
\end{aligned}$$

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

For linear isotropic media:

$$\mathbf{D}=\epsilon\mathbf{E}=\epsilon_0\mathbf{E}+\mathbf{P},\qquad \mathbf{H}=\frac{1}{\mu}\mathbf{B}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}.$$

$$c^2d\tau^2=c^2dt^2-dx^2-dy^2-dz^2=dx^\alpha\eta_{\alpha\beta}dx^\beta.$$

$$\eta_{\alpha\beta}=\begin{cases}+1 & \text{if } \alpha=\beta=0\\ -1 & \text{if } \alpha=\beta=1,2,3\\ 0 & \text{if } \alpha\neq\beta\end{cases}$$

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\nabla\right),\qquad \partial^\mu=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right).$$

$$\partial_\alpha F^{\alpha\beta}=\partial_\alpha\partial^\alpha A^\beta-\partial^\beta\partial_\alpha A^\alpha=\mu_0 j^\beta;\qquad F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0.$$

$$\|F^{\alpha\beta}\|=\begin{pmatrix}0 & -E^1/c & -E^2/c & -E^3/c\\ E^1/c & 0 & -B^3 & B^2\\ E^2/c & B^3 & 0 & -B^1\\ E^3/c & -B^2 & B^1 & 0\end{pmatrix}.$$

In spherical coordinates (r,θ,ϕ), with corresponding unit coordinate vectors $(\hat{\mathbf{r}},\hat{\theta},\hat{\phi})$, for a vector field $\mathbf{A}$ with components (A_r,A_θ,A_ϕ),

$$\nabla\times\mathbf{A}=\hat{\mathbf{r}}\,\tfrac{1}{r\sin\theta}\left(\tfrac{\partial}{\partial\theta}(A_\phi\sin\theta)-\tfrac{\partial A_\theta}{\partial\phi}\right)+\hat{\theta}\left(\tfrac{1}{r\sin\theta}\tfrac{\partial A_r}{\partial\phi}-\tfrac{1}{r}\tfrac{\partial}{\partial r}(rA_\phi)\right)+\hat{\phi}\tfrac{1}{r}\left(\tfrac{\partial}{\partial r}(rA_\theta)-\tfrac{\partial A_r}{\partial\theta}\right)$$

and for a scalar field $G(r,\theta,\phi)$

$$\nabla^2G=\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2G}{\partial\phi^2}.$$

End of Examination Paper **Prof WJ Spence**

Answer 1 (a) [5 marks] Since $\mathbf{E_{sc}}$ and $\mathbf{B_{sc}}$ are perpendicular, and similarily for the incident electric and magnetic fields, one has $|\mathbf{S_{sc}}| = |\mathbf{E_{sc}}|^2/c\mu_0$ and similarily for the incident flux. The time averaging factors cancel in the ratio and the result follows.

(b) [5 marks] The scatterer at $\mathbf{x}_j$ experiences the incident field with a phase factor differing from that at the origin by $e^{i\mathbf{k}_0\cdot\mathbf{x}_j}$. Its response will therefore also acquire this phase factor. Likewise the phase at the detector of the component scattered by this scatterer acquires a further factor $e^{-ik\mathbf{n}\cdot\mathbf{x}_j}$compared with what would have been received from a scatterer at the origin. So the phase of the contribution to $\mathbf{E}_{\mathrm{SC}}$ is modified by an overall factor $e^{i\mathbf{q}\cdot\mathbf{x}_j}$, so that the electric component of the scattered field is $\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}$ times what was the case for a single scatterer at the origin. Since the differential cross-section involves the square modulus of this, the result is as given, namely to multiply the result for a single scatterer by the structure factor.

(c) [5 marks] For N scatterers, the sum gives directly that $\mathcal{F}(0) = N^2$. For a large number of randomly-distributed scatterers, the phases of off-diagonal terms in the sum (obtained from expanding out the modulus squared) will cancel except close to the forward direction, provided that $|\mathbf{q}|a >> 1$. Then $\mathcal{F}(\mathbf{q}) \simeq N$.

(d) [5 marks] In a crystal, there are peaks in the structure function around $qa = 0, 2\pi, 4\pi, ...$, ie when the Bragg condition is satisfied, and then $\mathcal{F} = N^2$. The number of peaks is limited by the maximum value which qa can attain, $qa \leq 2ka$, so that at long wavelengths only the forward peak occurs. This has a width determined by $q \leq 2\pi/Na$, corresponding to scattering angles less than or of order λ/L, where L is the linear size of the crystal. (In each direction one finds $\mathcal{F}(\mathbf{q}) = \frac{\sin^2(Nqa/2)}{\sin^2(qa/2)}$; this formula is not required for full marks.)

Answer 2 (a) [5 marks] We have

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{g(1-\cos\theta)}{4\pi r}\right) = \frac{g}{4\pi r^2}\hat{\mathbf{r}},$$

where we have used the fact that rA_ϕ is independent of r for this potential.

(b) [3 marks] This is the magnetic field for a magnetic monopole of charge g, sited at the origin.

(c) [8 marks] Since

$$\mathbf{p}\cdot\dot{\mathbf{p}} = \frac{egm\gamma(v)}{4\pi r^3}\mathbf{v}\cdot(\mathbf{v}\times\mathbf{r}) = 0$$

using the cyclic identity for the triple product in the last step, it follows that $\dot{E} = 0$. Secondly,

$$\dot{\mathbf{J}} = \gamma(v)m\mathbf{v}\times\mathbf{v} + \frac{eg}{4\pi r^3}\mathbf{r}\times(\mathbf{v}\times\mathbf{r}) - \frac{eg}{4\pi}(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r^3}\mathbf{r}) = 0,$$

using the result $\dot{r} = \frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r}$. E is the energy of the electrically charged particle, and $\mathbf{J}$ is the angular momentum of the system, the first term being the orbital angular momentum of the electrically charged particle, and the second term the angular momentum in the electromagnetic fields.

(d) [4 marks] When $\mathbf{v} = 0$, then $|\mathbf{J}| = \frac{eg}{4\pi}$, and assuming that this is quantised in half-integral units of $\hbar$ (as is the case for intrinsic angular momentum), one derives the result.

Answer 3 (a) [3 marks] $k = \omega/c$, $\mathbf{r}$ is the vector from the dipole center to the field point, and $\mathbf{r} = r\mathbf{n}$, with $\mathbf{n}$ a unit vector.

(b) [5 marks] We have

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left(\frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r} \mathbf{n} \times \mathbf{m} \right).$$

The leading term at large r is

$$\frac{\mu_0}{4\pi r}(\nabla r) \times (\mathbf{n} \times \mathbf{m}) = -\frac{k^2 \mu_0}{4\pi r^2}(\mathbf{r} \times (\mathbf{n} \times \mathbf{m})).$$

(c) [5 marks] One has

$$\mathbf{S} = \frac{c}{\mu_0}|\mathbf{B}|^2\mathbf{n} = \frac{ck^4\mu_0}{16\pi^2 r^4}|\mathbf{r} \times (\mathbf{n} \times \mathbf{m})|^2\,\mathbf{n} = \frac{\mu_0\omega^4}{16\pi^2 r^2 c^3}|\mathbf{m}|^2 \sin^2\theta\,\mathbf{n},$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{m}$.

(d) [7 marks] Here

$$\frac{dE}{dt} = -\int \mathbf{S} \cdot d\mathbf{A} = -\frac{\mu_0\omega^4}{16\pi^2 c^3} m_\alpha^2 \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta\, d\theta\, d\phi,$$

where the surface integral is over the surface of a sphere which contains the star, and $m_\alpha = |\mathbf{m}|\sin\alpha$ is the length of the component of the dipole moment which oscillates as $e^{-i\omega t}$. The surface integral in the final expression in the equation above equals $\frac{8\pi}{3}$, whence the result follows.

Answer 4 (a) [4 marks] The retarded time is defined by the unique point to the past of the field point x on the trajectory of the particle from which an influence propagating at the speed of light reaches the position $\mathbf{x}$ at the time $ct = x^0$. The corresponding time r^0/c on the trajectory is called the retarded time, $t_{\rm ret}$, with $c(t - t_{\rm ret}) = R$. R is the distance from field point to particle, $R = |\mathbf{x} - \mathbf{r}(\tau_0)|$, with $R\mathbf{n} = \mathbf{x} - \mathbf{r}(\tau_0)$.

(b) [6 marks] Working to lowest order in $\boldsymbol{\beta}$ and $1/R$,

$$\begin{aligned}\mathbf{E}_{\mathbf{far}} &= -\dot{\mathbf{A}} - \nabla\Phi \\ &= -\frac{\mu_0 qc}{4\pi}(\frac{\dot{\beta}}{R} - \frac{\beta}{R^2}\dot{R}) \\ &= \frac{q}{4\pi\epsilon_0 cR}((\mathbf{n}\cdot\dot{\beta})\mathbf{n} - \dot{\beta})\end{aligned}$$

(where we take the expressions at the retarded time) giving the required result.

(c) [4 marks] We have

$$\mathbf{S}_{\mathbf{far}} = \frac{1}{\mu_0}\mathbf{E}_{\mathbf{far}} \times \mathbf{B}_{\mathbf{far}}$$

but $\mathbf{n}\cdot\mathbf{E}_{\mathbf{far}} = 0$ whence the result follows.

(d) [6 marks] The power radiated per unit solid angle is $\frac{dP}{d\Omega} = \frac{1}{\mu_0 c}|R\mathbf{E}_{\mathbf{far}}|^2$. Inserting the expression for $\mathbf{E}_{\mathbf{far}}$, gives $\frac{dP}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\dot{\mathbf{u}}^2\sin^2\theta$, where θ is the angle between the direction of the field point and the instantaneous acceleration $\dot{\mathbf{u}}$ of the particle. Integrating this over solid angles gives the required expression.

Answer 5 (a) [3 marks] We have $\partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu$ in Lorentz gauge, from which it is straightforward to deduce the two equations.

(b) [4 marks] This requires integrating the term involving $\frac{\partial^2}{\partial t^2}$ twice by parts to bring down a factor of $-\omega^2$ from the exponential, and dropping boundary terms assuming that the fields and their first two time derivatives fall off to zero at infinity.

(c) [3 marks] Here one pulls the d'Alembertian operator inside the integral and acts with it on G. This generates a delta function which is then integrated with $\mathbf{J}$ to give the required answer.

(d) [5 marks] The d'Alembertian operator is invariant under translations and spatial rotations, hence the function G must be a function of the scalar r alone. When $r \neq 0$, the delta function does not contribute and one has a standard second order ordinary differential equation for rG, which is solved by an arbitrary linear combination of the two exponentials.

(e) [5 marks] When $r \to 0$, then the $1/r$ term dominates on the left-hand side and one has

$$\frac{1}{r}\frac{d^2}{dr^2}(rG) = -4\pi\delta^3(\mathbf{r}).$$

This is Poisson's equation if one identifies $\Phi = G_k$, and $\rho = 4\pi\epsilon\delta^3(\mathbf{r})$. Using this in the given solution one finds that $G = 1/r$ and hence that one must have $A + B = 1$.