Mathematical Tripos 2004, Part IB : Electromagnetism, Lent 2004
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4 Electromagnetic induction

Recall the paragraph from Sec. 1.5, repeated here: The Maxwell equation

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1}$$

implies

$$\int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \nabla \wedge \mathbf{E} \cdot \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{2}$$

by applying Stokes's theorem to a fixed curve $C = \partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr}, \tag{3}$$

and the flux of $\mathbf{B}$ through (the open) surface S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{4}$$

then we get Faraday's Law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{5}$$

This will be studied now.

In chapter two we studied electric fields $\mathbf{E}$ such that

$$\nabla \wedge \mathbf{E} = 0, \quad \int_C \mathbf{E} \cdot \mathbf{dr} = 0, \tag{6}$$

called conservative, since, in virtue of $\nabla \wedge \mathbf{E} = 0$, there exists the electrostatic potential ϕ such that $\mathbf{E} = -\nabla\phi$. In chapter two it was assumed implicitly that there were no magnetic fields in the discussion, but it could equally have been assumed that we were dealing with non-conducting material (*e.g.* the vacuum or free space) and time-independent magnetic fields, since the latter would then be entirely uncoupled from the electrostatics.

Here we study time-dependent magnetic fields and the the non-conservative electric fields that accompany them. The latter may give rise to non-zero electromotive forces (or electromotances, or EMFs for short), and hence cause current flow.

We first make this study in the (pre-Maxwellian) approximation to the full Maxwell theory, in which

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}. \tag{7}$$

In other words, we neglect the displacement current, even though it was seen in Sec. 1.4 to be an essential ingredient of a consistent theory. It can be shown however that this is justified in the practically significant context in which there are alternating currents of low enough frequency flowing in media of high enough conductivity.

We look first at simple situations wherein it can be seen how time-dependent magnetic fields can produce non-zero EMFs and cause current flow.

4.1 Simple examples

If we talk about a bar magnet, we mean a piece of material in which the atomic spins, essentially small current loops, are all lined up, to produce a macroscopic magnetic moment, as in the left hand diagram.

A bar magnet moved relative to a fixed circuit, with a galvanometer, causes a current to flow in the circuit, as motion of the galvanometer needle indicates. There is current flow iff there bar magnet moves.

Suppose the bar magnet in this context is replaced by a second circuit, with a battery, and a current flowing, and with a movable part. Iff there is motion of the latter relative to the first circuit, then will the galvanometer record a current flow. (The magnetic field of the current in the first circuit does the business just as well as did the bar magnet.)

The permanent magnet set-up in the diagram produces magnetic fields in the curved slots in which the loop of a circuit can rotate. If the loop is made to rotate steadily, then an alternating current flows in the circuit. This is the principle of the (AC) generator.

The same set-up can be used to illustrate the principle of the electric motor. Across each slot there is a north and a south pole. Suppose the coil is lying with one side in each slot. When a current is passed through the coil, it flows in opposite directions on the two sides, so these feel equal and opposite forces. In other words a couple is being applied to the coil. If the shaft of the coil is free to rotate, the system can be coupled to pulleys or gears and do work.

4.2 Faraday's law of induction

Let C be either

(a) a fixed closed geometrical curve, or

(b) a physical, possibly moving circuit.

Let S be a surface bounded by $C = \partial S$.
Define the flux, of a possibly time-dependent magnetic field $\mathbf{B}$, through S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}. \tag{8}$$

Then Faraday's experimental law, valid in both the contexts (a) and (b), with an appropriate definition in each case of the EMF $\mathcal{E}$ in C, is

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{9}$$

In case (a)

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \boldsymbol{\nabla} \wedge \mathbf{E} \cdot \mathbf{dS} \tag{10}$$

and

$$\frac{d\Phi}{dt} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}. \tag{11}$$

Consistency of (9 –11) is now assured by means of the Maxwell equation (1), assumed true in general.

For case (b), consider the case of a physical circuit moving with velocity $\mathbf{v}$, possibly dependent on position and time, but $v \ll c$, in a time-dependent magnetic field $\mathbf{B}$.

The force on a particle of charge q moving with velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$, and therefore also in its accompanying electric field $\mathbf{E}$, is given by eq. (16) of Sec. 1.3:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}). \tag{12}$$

Hence we define the electromotance or EMF in C as

$$\mathcal{E} = \frac{1}{q} \oint_C \mathbf{F} \cdot \mathbf{dr} = \oint_C (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \mathbf{dr}. \tag{13}$$

We must show that, in context (b), (9) and (13) are compatible with the Maxwell equation (1).

To achieve this, we set out from an expression for $\frac{d\Phi}{dt}$

$$\frac{d\Phi}{dt} = \lim_{\delta t \to 0} \left[\frac{1}{\delta t} \left(\int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS'} - \int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} \right) \right]. \tag{14}$$

Then we apply the divergence theorem at time $(t + \delta t)$ to the spatial volume V bounded by S, S' and the curved surface Σ swept out by the circuit C as it moved from position S at time t to position S' at $(t + \delta t)$.

$$\begin{aligned} 0 &= \int_V \boldsymbol{\nabla} \cdot \mathbf{B} d\tau \\ &= \int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS'} - \int_S \mathbf{B}(\mathbf{r}, t + \delta t) \cdot \mathbf{dS} + \oint_C \mathbf{B}(\mathbf{r}, t + \delta t) \cdot (\mathbf{dr} \wedge \mathbf{v} \delta t). \end{aligned} \tag{15}$$

Here, as the right-hand diagram purports to justify, we have used

$$\mathbf{dS} \approx \mathbf{dr} \wedge \mathbf{v} \delta t, \tag{16}$$

on Σ. Since the third term of (15) is proportional to δt and hence already small, we may neglect δt in the arguments of $\mathbf{B}$ and of $\mathbf{v}$ in it, having already neglected the variation of $\mathbf{B}$ and $\mathbf{v}$ with position across Σ.

The second integral in (15) has the Taylor expansion

$$\int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} + \delta t \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS}. \tag{17}$$

These remarks allow us to write (15) as

$$0 = \int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS} - \int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} - \delta t \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS} + \delta t \oint_C \mathbf{dr} \cdot \mathbf{v} \wedge \mathbf{B}(\mathbf{r}, t). \tag{18}$$

Dividing by δt, we see the first two terms in (18) allow us to bring in $\frac{d\Phi}{dt}$ using (14). So we get

$$0 = \frac{d\Phi}{dt} - \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS} + \oint_C \mathbf{dr} \cdot \mathbf{v} \wedge \mathbf{B}(\mathbf{r}, t). \tag{19}$$

The first term here is related by (9) to $\mathcal{E}$, which is defined in the present context by (13). Hence

$$0 = \oint_C \mathbf{E} \cdot \mathbf{dr} + \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS}, \tag{20}$$

the $\mathbf{v}$-dependent terms having cancelled, so that consistency is assured by the Maxwell equation (1), just as in case (a).

The significance of the minus sign in the definition (5) of the EMF reflects Lenz's law, which states that any EMF induced in a circuit by a change of flux through it tends to oppose any EMF (*e.g.* due to a battery) that already exists in the circuit.

4.3 The Faraday experiment

In the set-up shown the crossbar LM can slide with negligible friction parallel to ON. The uniform time independent magnetic field $\mathbf{B} = (0, 0, B)$ points upwards from the plane of the page.

We shall neglect the resistance of the wire $QMN(\mathcal{E}_0)OLP$ The circuit $C = OLMN(\mathcal{E}_0)$ thus has resistance

$$R, \tag{21}$$

i.e. the resistance R of LM. Also, for large B and R, we neglect magnetic fields arising from any current flowing in the system. The initial conditions are

$$x = x_0, \quad \dot{x} = 0, \quad I = I_0 = \frac{\mathcal{E}_0}{R} \quad \text{at} \quad t = 0. \tag{22}$$

The Biot-Savart law tells us that the force $\delta\mathbf{F}$ acting on the element $\delta\mathbf{r} = \delta y\mathbf{j} = \delta y(0,1,0)$ of LM is given by

$$\delta\mathbf{F} = I\delta\mathbf{r}\wedge\mathbf{B} = I\delta y\, B\mathbf{j}\wedge\mathbf{k} = I\delta y\, B\mathbf{i}. \tag{23}$$

So the total force on LM is

$$\mathbf{F} = Ia\,B\mathbf{i}. \tag{24}$$

By Newton's second law, we have

$$m\ddot{x} = IaB. \tag{25}$$

We cannot assume that I is independent of t, so that we are not yet ready to try to solve (25).

When LM is at x, the flux of $\mathbf{B}$ through $\partial S = C$ is

$$\Phi = \int_S \mathbf{B}\cdot\mathbf{dS} = \text{constant} + B(ax), \tag{26}$$

so that the EMF induced in C in the circuit is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba\dot{x}. \tag{27}$$

It follows now that the total EMF in the circuit at time t is

$$\mathcal{E}_0 + \mathcal{E} = \mathcal{E}_0 - Ba\dot{x}, \tag{28}$$

and that

$$\mathcal{E}_0 - Ba\dot{x} = IR. \tag{29}$$

Eqs. (25) and (29) enable the time dependence of I and $\dot{x}$ to be calculated. In view of our neglect of various effects, we have a reasonably simple differential equation for $x(t)$

$$m\ddot{x}R = aB(\mathcal{E}_0 - Ba\dot{x}), \tag{30}$$

indeed soluble quite nicely for small t. This solution exhibits what is expected in general, that the induced EMF opposes the battery EMF, and the current in C is reduced. These are two aspects of Lenz's law.

Lenz's law is a special case of more general belief: le Châtelier's principle. This can be stated as follows: a physical system in a steady state reacts by opposing any change imposed on it from outside.

We neglected the magnetic field due to the current induced in C, which opposes the battery produced I_0. But (using the result (38) from Sec. 3.3 a), we see that the field due to the induced current in LM *e.g.* points downwards on the plane of the diagram, and opposes $\mathbf{B}$. This too exemplifies a Lenz view: flux change of one sign produces currents which create flux of the the opposite sign.

4.4 Coil rotating in a fixed magnetic field

Let C be a closed rectangular curve $PQRS$ of area A. Very thin conducting wire is wrapped N times around the curve C with free ends connected to some external circuit.

Suppose C can rotate rigidly about a fixed axis $\mathbf{j} = (0, 1, 0)$ with angular velocity ω in the presence of a uniform time-independent magnetic field $\mathbf{B} = (0, 0, B)$.

When the normal to the coil makes an angle $\theta = \omega t$ to $\mathbf{B}$ as shown, so that $\mathbf{n} = \cos\theta\mathbf{k} + \sin\theta\mathbf{i}$, then the flux of $\mathbf{B}$ through the coil is

$$\int \mathbf{B} \cdot \mathbf{dS} = N\mathbf{B} \cdot \mathbf{n}A = NB\cos\theta A. \tag{31}$$

Hence the EMF induced in the circuit is

$$\mathcal{E} = -\frac{d\Phi}{dt} = NBA\omega\sin\omega t. \tag{32}$$

If the coil has resistance R, then the current induced in the coil is

$$I = \frac{NBA\omega}{R}\sin\omega t. \tag{33}$$

Using (64) of Sec. 3.7, we know that the couple exerted on the circuit by the magnetic field is

$$\mathbf{G} = N\oint_C \mathbf{r}\wedge(I\mathbf{dr}\wedge\mathbf{B}). \tag{34}$$

It can be shown, with the aid of Stoke's theorem, that this can be caste into the form

$$\mathbf{G} = \mathbf{m}\wedge\mathbf{B}, \tag{35}$$

where the magnetic moment of the plane N-loop coil is given, using (54) of Sec. 3.5, by $\mathbf{m} = NIA\mathbf{n}$. Proof of (35) will be attached to the end of chapter 5. Evaluating (35) we find that $\mathbf{G} = -IANB\sin\theta\mathbf{j}$, which, in the spirit of Lenz's law, tends to counter the torque that applies the angular velocity to the coil.

4.5 Inductance and magnetic energy

We will illustrate these concepts by reference to the long solenoid of Sec. 3.3. First we recall the context and some of the results obtained there.

The solenoid has N turns of wire per unit length and length l very large so that end effects can be neglected. It carries current I. It is cylindrical with axis $\mathbf{k} = (0, 0, 1)$, and cross-sectional area A. The magnetic field due to the current flow is

$$\mathbf{B} = \mu_0 NI\mathbf{k} \tag{36}$$

inside the solenoid and zero outside. The flux of $\mathbf{B}$ through one turn of the solenoid is

$$\mu_0 NIA \tag{37}$$

and through all Nl turns is

$$\Phi = \mu_0 N^2 lIA. \tag{38}$$

This is proportional to I, and we define the (self)-inductance L of the coil via $\Phi = LI$ giving

$$L = \mu_0 N^2 l A = \mu_0 N^2 V, \tag{39}$$

where $V = Al$ is the volume of the solenoid.

Suppose now that the long solenoid, which remains stationary in this discussion, is attached to a battery of EMF $\mathcal{E}_0$, so that the total EMF in the circuit is

$$\mathcal{E}_0 + \mathcal{E}_{induced} = \mathcal{E}_0 - \frac{d\Phi}{dt} = \mathcal{E}_0 - L\dot{I}, \tag{40}$$

and Ohm's law reads

$$\mathcal{E}_0 = IR + L\dot{I}. \tag{41}$$

Now (*cf.* eq. (17) of Sec. 3.1), the work done by the battery in time δt is

$$\delta W = \mathcal{E}_0 I \delta t = RI^2 \delta t + LI\dot{I}\delta t \tag{42}$$

The first term corresponds to Ohmic heat generation. As it is not of immediate interest, we suppose the circuit is of negligible resistance and neglect it. Hence

$$\delta W = LI\dot{I}\delta t = \delta(\frac{1}{2}LI^2). \tag{43}$$

Assuming the magnetic energy W is zero at $t = 0$ when $I = 0$ also, we have

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi. \tag{44}$$

Next we develop the expression (44)

$$W = \frac{1}{2}I\Phi = \frac{1}{2}I\int_S \mathbf{B}.\mathbf{dS} = \frac{1}{2}I\int_S \boldsymbol{\nabla}\wedge\mathbf{A}.\mathbf{dS} = \frac{1}{2}I\oint_C \mathbf{A}.\mathbf{dr}. \tag{45}$$

Hence we can pass in now familiar fashion to the magnetic energy of a continuous distribution of current density $\mathbf{J}$ confined to a finite region region of space near to which we take our origin.

$$W = \frac{1}{2}\int_V \mathbf{J}.\mathbf{A}d\tau, \quad V = \text{all space}. \tag{46}$$

This leads to an important alternative expression for the magnetic energy.

$$W = \frac{1}{2\mu_0}\int_V \mathbf{A}.\boldsymbol{\nabla}\wedge\mathbf{B}d\tau = \frac{1}{2\mu_0}\int_V \left[-\boldsymbol{\nabla}.(\mathbf{A}\wedge\mathbf{B}) + \mathbf{B}.(\boldsymbol{\nabla}\wedge\mathbf{A})\right] d\tau. \tag{47}$$

We may use the divergence theorem on the first term and see that it vanishes provided that (as can be checked) its integrand goes to zero fast enough as r goes to infinity. Now from (47) we get the important result

$$W = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau. \tag{48}$$

It can easily be seen, using (39) for L and (36) for $B = |\mathbf{B}|$, that the two expressions (44) and (48) for the magnetic energy W give the same result in the case of the long solenoid

$$\frac{1}{2}LI^2 = \frac{1}{2}(\mu_0 N^2 V)I^2 = \frac{1}{2\mu_0}B^2 V = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau. \tag{49}$$